Assam Academy of Mathematics
Assam Mathematics Olympiad 2024
Category III (Classes IX - XI)
1st September 2024

Full marks : 100 Time : 3 hours

There are 18 questions. Questions 1 to 5 carry 2 marks each. Questions 6 to 13 carry 5 marks each. Questions 14 to 18 carry 10 marks each.

ইয়াত 18 টা প্ৰশ্ন আছে। 1 ৰ পৰা 5 লৈ প্ৰতিটো প্ৰশ্নত 2 নম্বৰকৈ আছে। 6 ৰ পৰা 13 লৈ প্ৰতিটো প্ৰশ্নত 5 নম্বৰকৈ আছে। আৰু 14 ৰ পৰা 18 লৈ প্ৰতিটো প্ৰশ্নত 10 নম্বৰকৈ আছে।

There may be various other ways of solutions than those shown here. Queries or suggestions regarding the solutions can be mailed to mail@aamonline.in
ইয়াত দেখুওৱা ধৰণবিলাকৰ বাহিৰেও প্ৰশ্নবোৰৰ সমাধানৰ আন বিভিন্ন উপায় থাকিব পাৰে। সমাধানবোৰৰ বিষয়ে কিবা প্ৰশ্ন বা পৰামৰ্শ থাকিলে mail@aamonline.in লৈ মেইল কৰিব পাৰে।

1. How many 4 digit even numbers have all digits distinct ?

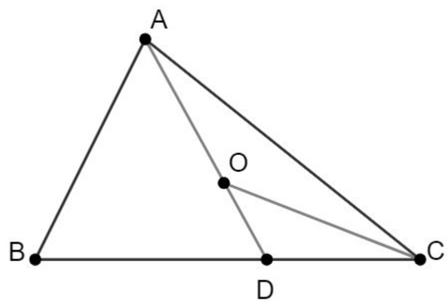
   4 টা অংকবিশিষ্ট এনেকুৱা কিমানটা যুগ্ম সংখ্যা আছে যাৰ অংকসমূহ পৃথক পৃথক?

   Ans : If it ends in 0 there are $9 \times 8 \times 7$ options for the first, second and third digits. Otherwise there are 4 options for the last digit, 8 options for the first digit (it can't be 0 or the same as the last digit), 8 options for the second digit and 7 options for the third digit. So there are $9 \times 8 \times 7 + 8 \times 8 \times 7 \times 4 = 2296$ such numbers.

2. Point O is inside the triangle ABC. Prove that $AO + OC < AB + BC$.

   O বিন্দুটো ত্ৰিভুজ ABC ৰ ভিতৰত আছে। প্ৰমাণ কৰা যে $AO + OC < AB + BC$।

   Ans : We know sum of two sides of a triangle is greater than the third side. We join AO and extend it to meet BC at D. We join OC.



   In $\triangle ABD$, $AB + BD > AD$ i.e. $AB + BD > AO + OD$. In $\triangle ODC$, $OD + DC > OC$. Adding, $AB + BD + OD + DC > AO + OD + OC$ which gives $AB + BC > AO + OC$.

3. Find all integers n such that $n^2 + 1$ is divisible by $n + 1$.

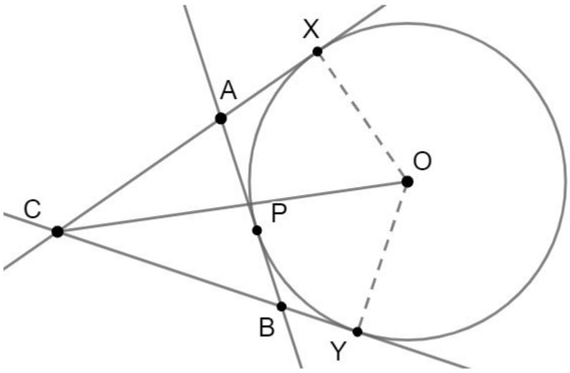
   এনেকুৱা আটাইবোৰ অখণ্ড সংখ্যা n নিৰ্ণয় কৰা যাতে $n^2 + 1$ সংখ্যাটো $n + 1$ ৰে বিভাজ্য হয়।

   Ans : We know $(n+1)|(n^2-1)$. Given, $(n+1)|(n^2+1)$. So, $(n+1)|(n^2+1)-(n^2-1) = 2$. So, $n + 1 = 1, 2, -1, -2$ i.e. $n = 0, 1, -2, -3$.

4. Consider a circle with radius 5 and centre O. Two tangents are drawn from an external point C to the circle at points X and Y on the circumference. A and B are points on CX and CY respectively such that AB is a tangent at P. If CO = 13, find the perimeter of Δ ABC.

5 একক ব্যাসাৰ্ধ আৰু O কেন্দ্ৰবিশিষ্ট বৃত্ত এটা বিবেচনা কৰা। বৃত্তৰ বাহিৰৰ এটা বিন্দু C ৰ পৰা ইয়াৰ পৰিধিত থকা দুটা বিন্দু X আৰু Y লৈ দুডাল স্পৰ্শক অঁকা হ'ল। A আৰু B হৈছে যথাক্ৰমে CX আৰু CY ৰ ওপৰত থকা এনেকুৱা দুটা বিন্দু যাতে P বিন্দুত AB এডাল স্পৰ্শক হয়। যদি $CO = 13$ একক, তেন্তে ΔABC ৰ পৰিসীমা নিৰ্ণয় কৰা।

Ans : Given $CO = 13$ and radius is 5. So, $OX = 5$. In right angled triangle CXO, $CX = \sqrt{CO^2 - OX^2} = \sqrt{13^2 - 5^2} = 12$. Again, since lengths of tangents from an external point to a circle are equal, so $CX = CY = 12$. Also, $AP = AX$ and $BP = BY$. So, perimeter of $\triangle ABC$ is $CA+AB+CB = CA+AP+PB+CB = CA+AX+BY+CB = CX+CY = 12+12 = 24$.



5. Let $S(r)$ denotes the sum of the geometric series

$$12 + 12r + 12r^2 + 12r^3 + \cdots\cdots, -1 < r < 1$$

Let $S(a)S(-a) = 2024$ for $-1 < a < 1$. Find $6(S(a) + S(-a))$.

ধৰা হ'ল $S(r)$ য়ে

$$12 + 12r + 12r^2 + 12r^3 + \cdots\cdots, -1 < r < 1$$

এই গুণোত্তৰ শ্ৰেণীটোৰ যোগফলক সূচায়। ধৰা হ'ল $S(a)S(-a) = 2024$, $-1 < a < 1$। তেন্তে $6(S(a)+S(-a))$ ৰ মান নিৰ্ণয় কৰা।

Ans :

$$2024 = S(a)S(-a) = \frac{12}{1-a}.\frac{12}{1-(-a)}$$
$$\Rightarrow 1 - a^2 = \frac{12 \times 12}{2024} = \frac{18}{253}$$

Therefore

$$S(a) + S(-a) = \frac{12}{1-a} + \frac{12}{1-(-a)} = \frac{12[(1-a)+(1+a)]}{1-a^2} = \frac{1012}{3}.$$

So, $6(S(a) + S(-a)) = 2024$.

6. If the equation $x^6+ax^5+bx^4+cx^3+dx^2+ex+499=0$ has positive integer roots, find the value of a.

যদি $x^6+ax^5+bx^4+cx^3+dx^2+ex+499=0$ সমীকৰণটোৰ ধনাত্মক অখণ্ড মূল থাকে, তেন্তে aৰ মান নিৰ্ণয় কৰা।

Ans : The equation is of degree 6. The product of the six roots is 499 which is a prime number. Since the roots are positive integers, so one root is 499 and the other 5 roots are 1. Hence, $-a$= Sum of roots = $1+1+1+1+1+499=504$. So, $a=-504$.

7. Solve : $(5+2\sqrt{6})^{x^2-5}+(5-2\sqrt{6})^{x^2-5}=10$.

সমাধান কৰা: $(5+2\sqrt{6})^{x^2-5}+(5-2\sqrt{6})^{x^2-5}=10$.

Ans : Observe that $5-2\sqrt{6}=\dfrac{(5-2\sqrt{6})(5+2\sqrt{6})}{5+2\sqrt{6}}=\dfrac{1}{5+2\sqrt{6}}$

$$\begin{aligned}
&(5+2\sqrt{6})^{x^2-5}+(5-2\sqrt{6})^{x^2-5}=10\\
\Rightarrow&(5+2\sqrt{6})^{x^2-5}+\left(\frac{1}{5+2\sqrt{6}}\right)^{x^2-5}=10\\
\Rightarrow&\left((5+2\sqrt{6})^{x^2-5}\right)^2-10(5+2\sqrt{6})^{x^2-5}+1=0\\
\Rightarrow&\left((5+2\sqrt{6})^{x^2-5}\right)^2-2\times5\times(5+2\sqrt{6})^{x^2-5}+25-24=0\\
\Rightarrow&\left((5+2\sqrt{6})^{x^2-5}-5\right)^2=24=(2\sqrt{6})^2\\
\Rightarrow&(5+2\sqrt{6})^{x^2-5}=5\pm2\sqrt{6}
\end{aligned}$$

Thus, either $(5+2\sqrt{6})^{x^2-5}=5+2\sqrt{6}$ which gives $x^2-5=1$ i.e. $x=\pm\sqrt{6}$ or we have $(5+2\sqrt{6})^{x^2-5}=5-2\sqrt{6}$ which gives $x^2-5=-1$ i.e. $x=\pm2$.

8. Find the number of functions from $S=\{1,2,3,\ldots,20\}$ to itself such that $f(n)$ is a multiple of 3 whenever n is a multiple of 4. How many of these functions are one one ?

$S=\{1,2,3,\ldots,20\}$ সংহতিটোৰ পৰা নিজলৈকে (S লৈকে) এনেকুৱা কিমানটা ফলন থাকিব যাতে n টো, 4 ৰ গুণিতক হ'লে $f(n)$ টো, 3 ৰ এটা গুণিতক হয়। এই ফলনসমূহৰ ভিতৰত কিমানটা একৈকী?

Ans : There are 5 multiples of 4 in and 6 multiples of 3 in $S=\{1,2,3,\ldots,20\}$. The multiples of 4 can be mapped to the multiples of 3 in 6^5 ways. Remaining 15 numbers can be mapped to any of the 20 numbers in 20^{15}. So number of functions is $6^5\times20^{15}$. If the functions are one one then the 5 multiples of 4 can be mapped to the 6 multiples of 3 in $6\times5\times4\times3\times2=6!$ ways. The remaining 15 numbers can be mapped with 15 numbers in $15!$ ways. So, the number of one one functions is $6!\times15!$.

9. If a,b,c are positive real numbers, show that

$(a^3b+b^3c+c^3d+d^3a)(ab^3+bc^3+cd^3+da^3)\geq16a^2b^2c^2d^2$.

যদি a,b,c ধনাত্মক বাস্তৱ সংখ্যা, তেন্তে দেখুওৱা যে

$(a^3b+b^3c+c^3d+d^3a)(ab^3+bc^3+cd^3+da^3)\geq16a^2b^2c^2d^2$।

Ans : Let $a^3b = x_1^2$, $b^3c = x_2^2$, $c^3d = x_3^2$, $d^3a = x_4^2$ and $ab^3 = y_1^2$, $bc^3 = y_2^2$, $cd^3 = y_3^2$, $da^3 = y_4^2$.

$$\begin{aligned}
&(a^3b + b^3c + c^3d + d^3a)(ab^3 + bc^3 + cd^3 + da^3) \\
&= (x_1^2 + x_2^2 + x_3^2 + x_4^2)(y_1^2 + y_2^2 + y_3^2 + y_4^2) \\
&\geq (x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4)^2 \text{ (By Cauchy Schwartz inequality)} \\
&= (a^2b^2 + b^2c^2 + c^2d^2 + d^2a^2)^2 \\
&\geq (4(a^2b^2 \times b^2c^2 \times c^2d^2 \times d^2a^2)^{1/4})^2 \text{ (By AM GM inequality)} \\
&= 16a^2b^2c^2d^2
\end{aligned}$$

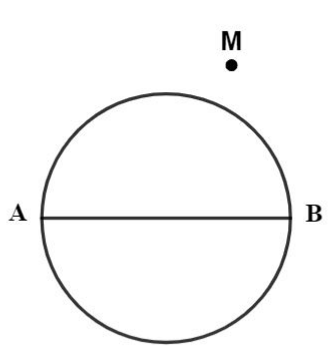
10. Find all prime numbers p such that $16p + 1$ is a perfect cube.

এনেকুৱা আটাইবোৰ মৌলিক সংখ্যা p বিচাৰি উলিওৱা যাতে $16p + 1$ এটা পূৰ্ণঘন সংখ্যা হয়।

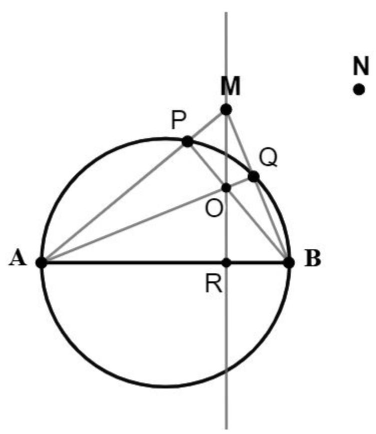
Sol. Let $16p+1 = q^3$. So, $16p = (q-1)(q^2+q+1)$. We observe that (q^2+q+1) is always odd irrespective of whether q is odd or even. So, 16 must be a factor of $(q-1)$ and p must be a multiple of (q^2+q+1). But p is prime. So $p = q^2+q+1$ and $16 = q-1$, which gives $q = 17$ and $p = 17^2 + 17 + 1 = 307$. Hence, $p = 307$ (a prime number) is the only possible value.

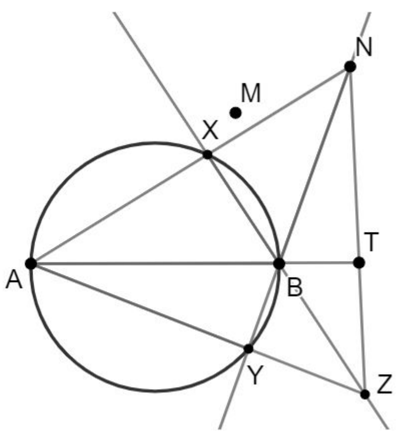
11. Consider a circle with diameter AB. M and N are points as shown in the figure. Using only a pencil and a ruler construct a perpendicular from M to AB and a perpendicular from N to AB extended. You are not allowed to use compass, protractor or any other angle measuring device. Write all the steps of the construction and draw the corresponding diagrams. Justify your construction with geometrical explanation.

AB ব্যাসযুক্ত বৃত্ত এটা বিবেচনা কৰা। চিত্ৰত দেখুওৱা ধৰণে M আৰু N দুটা বিন্দু। কেৱল পেঞ্চিল আৰু মাপনী ব্যৱহাৰ কৰি M ৰ পৰা AB লৈ এডাল লম্ব আৰু N ৰ পৰা বৰ্ধিত AB লৈ এডাল লম্ব অংকন কৰা। কম্পাছ, কোণমান যন্ত্ৰ বা আন কোনোধৰণৰ কোণ জোখা যন্ত্ৰ ব্যৱহাৰ নকৰিবা। তোমাৰ অংকনৰ যুক্তিযুক্ততা ব্যাখ্যা কৰা।



Ans : We join AM which intersects the circle at P. We join BM which intersects the circle at Q. We join AQ and BP which intersect each other at O. We join MO and extend it to meet the diameter AB at R. Now, $\angle APB = \angle AQB = 90°$.





Thus, AQ and BP are altitudes of $\triangle MAB$. But altitudes of a triangle are concurrent. Since AQ, BP and MOR are concurrent so MOR must be the altitude through M. So, MOR is perpendicular to AB.

Similar to the first case, we join AN which intersects the circle at X. We join BN which when extended intersects the circle at Y. We join AY and BX which when extended intersect each other at Z. We join NZ which meets AB extended at T. Now, $\angle AYB = \angle AXB = 90°$.

Thus, AY and BX are altitudes of $\triangle NAB$. But altitudes of a triangle are concurrent. Since AYZ, ZBX and NTZ are concurrent so NTZ must be the altitude through N. So, NTZ is perpendicular to AB extended.

12. How many sets of 3 numbers each can be formed from the first twenty natural numbers if no two consecutive numbers are to be in a set ?

প্ৰথম বিশটা স্বাভাৱিক সংখ্যাৰ পৰা এনেকুৱা কিমানটা সংহতি গঠন কৰিব পৰা যাব যাতে প্ৰতিটো সংহতিত তিনিটাকৈ সংখ্যা থাকে আৰু কোনো সংহতিতেই দুটা ক্ৰমিক সংখ্যা নাথাকে?

Ans : Three numbers can be chosen from twenty numbers in $\binom{20}{3} = 1140$ ways. But out of these selections, there are some with two or three consecutive numbers. We need to exclude these. There are 19 pairs of consecutive numbers i.e. $(1, 2), (2, 3), \ldots\ldots, (19, 20)$. So, two consecutive numbers can appear in 19 ways and for each such appearance, the third number can be chosen in 18 ways. So, we exclude $19 \times 18 = 342$. But in this process, sets with three consecutive numbers have been excluded twice. There are 18 such sets. So, we include them once again. Thus, the required number of such sets is $1140 - 342 + 18 = 816$.

13. Prove that for every positive integer n, the number $5^n + 2 \times 3^{n-1} + 1$ is a multiple of 8.

প্ৰমাণ কৰা যে প্ৰতিটো ধনাত্মক অখণ্ড সংখ্যা nৰ বাবে, $5^n + 2 \times 3^{n-1} + 1$ সংখ্যাটো 8ৰ গুণিতক হয়।

Ans : For $n = 1$, $5^n + 2 \times 3^{n-1} + 1 = 8$ which is a multiple of 8. Assume the induction hypothesis that $5^k + 2 \times 3^{k-1} + 1$ is a multiple of 8 for some $k \geq 1$. Then,

$$\begin{aligned} 5^{k+1} + 2 \times 3^k + 1 &= 5 \times 5^k + 3 \times 2 \times 3^{k-1} + 1 \\ &= 5(5^k + 2 \times 3^{k-1} + 1) - 2 \times 2 \times 3^{k-1} - 4 \\ &= 5 \times \text{a multiple of 8} - 4(3^{k-1} + 1) \\ &= 5 \times \text{a multiple of 8} - 4 \times \text{an even number} \\ &= \text{Mutiple of 8} \end{aligned}$$

Hence, the result follows by induction.

14. Consider numbers of the form $p^2 - q^2$ where p and q are primes greater than 3. Show that 24 is a common divisor of all such numbers. Is it the greatest such common divisor ? Justify.

$p^2 - q^2$ আকাৰৰ সংখ্যাসমূহ বিবেচনা কৰা য'ত p আৰু q হৈছে 3 তকৈ ডাঙৰ মৌলিক সংখ্যা। দেখুওৱা যে এই আকাৰৰ সকলোবোৰ সংখ্যাৰ এটা সাধাৰণ উৎপাদক হৈছে 24। এইটো তেনেকুৱা গৰিষ্ঠ সাধাৰণ উৎপাদক হয়নে? যুক্তি প্ৰদৰ্শন কৰা।

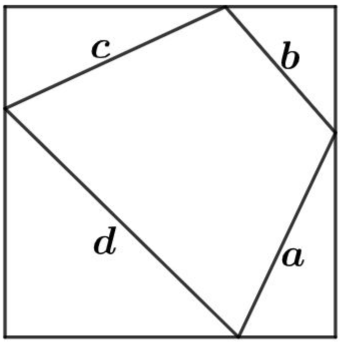
Ans : $p^2 - q^2 = (p - q)(p + q)$. Since p and q are primes greater than 3, so they are odd. So, $p - q$ and $p + q$ are both even i.e. $p^2 - q^2$ is a multiple of 4. We claim that at least one of $p - q$ and $p + q$ is a multiple of 4. Suppose that none of them is a multiple of 4. So, both will be of the form $4k + 2$. In that case, their sum will be divisible by 4 but that is not possible as the sum is $2p$ and p is odd. Hence, one of them is divisible by 4 so that $p^2 - q^2$ is divisible by 8.

Next, we show that $p^2 - q^2$ is also divisible by 3. Since p and q are primes greater than 3, so none of them is divisible by 3. So they are of the form $3k + 1$ or $3k + 2$. If both are of the form $3k + 1$, then $p - q$ is divisible by 3. If both are of the form $3k + 2$, then $p - q$ is divisible by 3. If one is of the form $3k+1$ and the other is of the form $3k+2$, then $p+q$ is divisible by 3. So, in any case p^2-q^2 is divisible by 3.

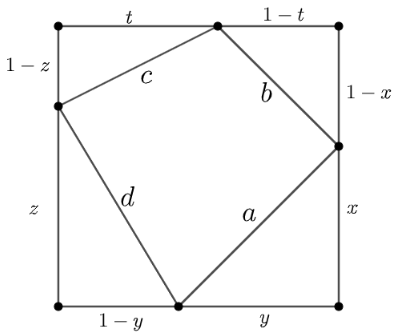
Hence, 24 is a common divisor of the numbers of the form p^2-q^2. Also, 24 is the greatest such common divisor as $7^2-5^2=24$.

15. A point is chosen on each side of a unit square (a square with all sides equal to 1). These four points form a quadrilateral with sides a,b,c,d. Show that $2\le a^2+b^2+c^2+d^2\le 4$ and $2\sqrt{2}\le a+b+c+d\le 4$.

এটা একক বৰ্গৰ (এনেকুৱা এটা বৰ্গ যাৰ সকলোবোৰ বাহুৰ দীঘ 1 একক) প্ৰতিটো বাহুৰ ওপৰত একোটাকৈ বিন্দু বাচি লোৱা হ'ল। এই বিন্দু চাৰিটাই এটা চতুৰ্ভুজ গঠন কৰিব যাৰ বাহুসমূহৰ দীঘ হৈছে a,b,c,d। দেখুওৱা যে $2\le a^2+b^2+c^2+d^2\le 4$ আৰু $2\sqrt{2}\le a+b+c+d\le 4$।



Ans : We introduce the lengths x,y,z,t as follows.



By Pythagoras theorem,

$$\begin{aligned}a^2&=x^2+y^2\\ b^2&=(1-t)^2+(1-x)^2\\ c^2&=t^2+(1-z)^2\\ d^2&=z^2+(1-y)^2\end{aligned}$$

$$\begin{aligned}a^2+b^2+c^2+d^2&=(1+2x^2-2x)+(1+2y^2-2y)+(1+2z^2-2z)+(1+2t^2-2t)\\ &=\left\{\frac{1}{2}+2\left(x^2-x+\frac{1}{4}\right)\right\}+\left\{\frac{1}{2}+2\left(y^2-y+\frac{1}{4}\right)\right\}\\ &+\left\{\frac{1}{2}+2\left(z^2-z+\frac{1}{4}\right)\right\}+\left\{\frac{1}{2}+2\left(t^2-t+\frac{1}{4}\right)\right\}\\ &=2\left(x-\frac{1}{2}\right)^2+2\left(y-\frac{1}{2}\right)^2+2\left(z-\frac{1}{2}\right)^2+2\left(t-\frac{1}{2}\right)^2+2\end{aligned}$$

Now, $0 \le x, y, z, t \le 1$, so that $-\frac{1}{2} \le x - \frac{1}{2}, y - \frac{1}{2}, z - \frac{1}{2}, t - \frac{1}{2} \le \frac{1}{2}$. Thus,

$0 \le \left(x - \frac{1}{2}\right)^2 \le \frac{1}{4}$, $0 \le \left(y - \frac{1}{2}\right)^2 \le \frac{1}{4}$, $0 \le \left(z - \frac{1}{2}\right)^2 \le \frac{1}{4}$ and $0 \le \left(t - \frac{1}{2}\right)^2 \le \frac{1}{4}$.

It then follows that $2 \le a^2 + b^2 + c^2 + d^2 \le 4$.

Again, by triangle inequality, $a \le x + y$, $b \le (1-x) + (1-t)$, $c \le (1-z) + t$, $d \le z + (1-y)$. Adding, $a + b + c + d \le 4$.

Observe that from any real numbers m, n, we have

$(m+n)^2 = m^2 + n^2 + 2mn \le m^2 + n^2 + m^2 + n^2$ as $2mn \le m^2 + n^2$.

This gives $m^2 + n^2 \ge \dfrac{(m+n)^2}{2}$

So,

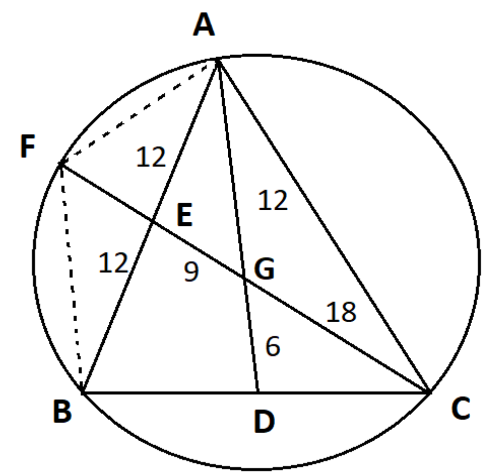
$$\begin{aligned} a+b+c+d &= \sqrt{x^2+y^2} + \sqrt{(1-t)^2+(1-x)^2} + \sqrt{t^2+(1-z)^2} + \sqrt{z^2+(1-y)^2} \\ &\ge \sqrt{\frac{(x+y)^2}{2}} + \sqrt{\frac{(1-t+1-x)^2}{2}} + \sqrt{\frac{(t+1-z)^2}{2}} + \sqrt{\frac{(z+1-y)^2}{2}} \\ &= \frac{1}{\sqrt{2}}(x+y+1-t+1-x+t+1-z+z+1-y) = \frac{4}{\sqrt{2}} = 2\sqrt{2} \end{aligned}$$

Thus $2\sqrt{2} \le a + b + c + d \le 4$.

16. In ΔABC, the medians AD and CE have lengths 18 and 27, respectively and $AB = 24$. Extend CE to intersect the circumcircle of ΔABC at F. Find the area of ΔAFB.

ΔABC ত AD আৰু CE মধ্যমাৰ দীঘ হৈছে ক্ৰমে 18 আৰু 27। CE ক এনেভাৱে বৰ্ধিত কৰা হ'ল যাতে ই ΔABCৰ পৰিবৃত্তক F বিন্দুত ছেদ কৰে। ΔAFB ৰ কালি নিৰ্ণয় কৰা।

Sol.



Since the centroid G divides the median in the ratio $2:1$. So $AG = 12$, $GD = 6$, $CG = 18$ and $GE = 9$. Also, AD and CE are medians, so $BD = DC = \frac{BC}{2}$ and $AE = EB = 12$.

Now from the intersecting chord theorem,

$$\begin{aligned} EA \times EB &= EC \times EF \\ \Rightarrow EF &= \frac{12 \times 12}{27} \\ \Rightarrow EF &= \frac{16}{3}. \end{aligned}$$

Now,

$$\begin{aligned} (\Delta AEG) &= \frac{1}{2} \times 9 \times \sqrt{12^2 - \frac{9}{2}^2} \\ &= \frac{27}{4}\sqrt{55} \end{aligned}$$

Also,

$$\begin{aligned} \frac{\Delta AEF}{\Delta AEG} &= \frac{FE}{EG} \\ &= \frac{\frac{16}{3}}{9} \\ &= \frac{16}{27} \\ \Delta AEF &= \frac{16}{27} \times \frac{27}{4}\sqrt{55} \\ &= 4\sqrt{55}. \end{aligned}$$

But Fe is a median of ΔFAB. Therefore,

$$(\Delta FAB) = 2(\Delta AEF) = 8\sqrt{55}.$$

17. For each natural number n, let $p(n)$ denote the product of the digits of n. Find all n such that $p(n) = n^2 - 10n - 22$.

(Hint : Show that $p(n) \leq n$ for all n.)

প্ৰতিটো স্বাভাৱিক সংখ্যা n ৰ বাবে, ধৰা হ'ল $p(n)$ য়ে n ৰ অংককেইটাৰ পূৰণফলক সূচায়। এনেকুৱা আটাইবোৰ n নিৰ্ণয় কৰা যাতে $p(n) = n^2 - 10n - 22$।

(ইংগিত: দেখুওৱা যে সকলো n ৰ বাবে, $p(n) \leq n$।)

Ans : Suppose that n has exactly k digits. Then $n = 10^{k-1}a + b$, where a is the first digit and b has $k-1$ digits. Since each digit can be at most 9, so $p(n) \leq 9^{k-1} \cdot a \leq 10^{k-1} \cdot a \leq 10^{k-1} \cdot a + b = n$. So, $p(n) \leq n$ i.e. $n^2 - 10n - 22 < n$ i.e. $n^2 - 11n - 22 < 0$ i.e. $(n - \frac{11+\sqrt{209}}{2})(n - \frac{11-\sqrt{209}}{2}) < 0$. Thus, we have $\frac{11-\sqrt{209}}{2} < n < \frac{11+\sqrt{209}}{2} < \frac{11+15}{2} = 13$.

Now, observe that $p(n) \neq 0$ as $n^2 - 10n - 22 = 0$ does not have integer roots. The roots are $\frac{10 \pm \sqrt{188}}{2}$ i.e. $5 \pm \sqrt{47}$. So, $p(n) > 0$. This gives $n^2 - 10n - 22 > 0$ i.e. $(n - (5 + \sqrt{47}))(n - (5 - \sqrt{47})) > 0$ i.e. $n > 5 + \sqrt{47} > 11$.

Hence, $11 < n < 13$. So, the only possible n is 12. It is easy to verify that $12^2 - 10 \times 12 - 22 = 144 - 120 - 22 = 2 = p(12)$.

18. Consider a regular hexagon. Draw all its diagonals. So you have 15 line segments joining the 6 vertices. Colour each side and diagonal either blue or red. (You may use dotted lines to

distinguish if you do not have colours.) Show that no matter how you colour, there always exists a triangle determined by the vertices which has all sides blue or all sides red.

এটা সুষম ষড়ভুজৰ কথা বিবেচনা কৰা। ইয়াৰ আটাইবোৰ কৰ্ণ অংকন কৰা। এতিয়া তোমাৰ ওচৰত 6 টা শীৰ্ষবিন্দু সংযোগকাৰী 15 ডাল ৰেখাখণ্ড আছে। প্ৰতিটো বাহু আৰু প্ৰত্যেকডাল কৰ্ণক নীলা বা ৰঙা ৰং কৰা। (যদি তোমাৰ ওচৰত ৰং নাই, তেন্তে তুমি ফুট ফুট ৰেখাও ব্যৱহাৰ কৰিব পাৰা।) দেখুওৱা যে যেনেদৰে ৰং কৰিলেও, শীৰ্ষবিন্দুসমূহৰ পৰা গঠন হোৱা এনেকুৱা এটা ত্ৰিভুজ থাকিব যাৰ গোটেইকেইটা বাহুৰ ৰং নীলা বা ৰঙা হ'ব।

Ans : One possible colour combination is shown here. We can see that there are several triangles which have all sides blue or all sides red. We now show that this always happens. Since all possible lines are drawn, so from each vertex there will be 5 line segments. Consider one such vertex A. Then by Pigeonhole principle, since 5 line segments are coloured using two colours, so at least three of the line segments will have the same colour. Without loss of generality, we assume AB, AC and AD are coloured blue. Then if we are to avoid blue triangle, BC must be red and CD must be red. Now, if BD is given blue colour, then ABD becomes a blue triangle and if BD is given red colour then BCD becomes a rd triangle. Thus, it is not possible to have any colour combination without at least one red or blue triangle.