Assam Academy of Mathematics
Assam Mathematics Olympiad 2023
Category III (Classes IX - XI)

Full marks : 100 | Time : 3 hours

There are 18 questions. Questions 1 to 5 carry 2 marks each. Questions 6 to 13 carry 5 marks each. Questions 14 to 18 carry 10 marks each.
ইয়াত 18 টা প্ৰশ্ন আছে। 1 ৰ পৰা 5 লৈ প্ৰতিটো প্ৰশ্নত 2 নম্বৰকৈ আছে। 6 ৰ পৰা 13 লৈ প্ৰতিটো প্ৰশ্নত 5 নম্বৰকৈ আছে। আৰু 14 ৰ পৰা 18 লৈ প্ৰতিটো প্ৰশ্নত 10 নম্বৰকৈ আছে।

There may be various other ways of solutions than those shown here. Queries or suggestions regarding the solutions can be mailed to mail@aamonline.in
ইয়াত দেখুওৱা ধৰণবিলাকৰ বাহিৰেও প্ৰশ্নবোৰৰ সমাধানৰ আন বিভিন্ন উপায় থাকিব পাৰে। সমাধানবোৰৰ বিষয়ে কিবা প্ৰশ্ন বা পৰামৰ্শ থাকিলে mail@aamonline.in লৈ মেইল কৰিব পাৰে।

1. What is the 288th term of the sequence

   $a, b, b, c, c, c, d, d, d, d, e, e, e, e, e, f, f, f, f, f, f, \ldots$?

   $a, b, b, c, c, c, d, d, d, d, e, e, e, e, e, f, f, f, f, f, f, \ldots$

   এই অনুক্ৰমটোৰ ২৮৮তম পদটো কি?

   Ans : The 1st letter a appears once, the 2nd letter b appears twice, the third letter c appears three times and so on. The nth letter appears n times consecutively. The nth letter thus goes upto $1+2+3+\ldots+n = \dfrac{n(n+1)}{2}$th term. Thus, for the given question $\dfrac{n(n+1)}{2} \leq 288$ i.e. $n(n+1) \leq 24^2$. This gives $n+1 \leq 24$ i.e. $n \leq 23$. The 23rd letter last occurs in the $\dfrac{23(23+1)}{2} =$ 276th term. And the 24th letter last occurs in the $\dfrac{24(24+1)}{2} = 300$th term. Thus, the 288th term is the 24th letter i.e. x.

2. An umbrella seller has umbrellas of 7 different colours. He has a total of 2023 umbrellas in stock but because of the plastic packaging, the colours are not visible. What is the minimum number of umbrellas that one must buy in order to ensure that at least 23 umbrellas are of the same colour ?

   এগৰাকী ছাতি বিক্ৰেতাৰ ওচৰত ৭টা পৃথক ৰঙৰ ছাতি আছে। তেওঁৰ ওচৰত মুঠ ২০২৩টা ছাতি মজুত আছে, কিন্তু প্লাষ্টিক পেকেজিঙৰ বাবে, বাহিৰৰ পৰা ৰঙবোৰ দেখা নাযায়। একে ৰঙৰ ২৩ টা ছাতি কিনিবলৈ হ'লে কোনো এজন মানুহে ন্যূনতম কিমানটা ছাতি কিনিব লাগিব?

   Ans : If there are at most 22 umbrellas of each colour, then one has bought at most $7 \times 22 = 154$ umbrellas. In order to ensure at least 23 umbrellas of the same colour, one has to buy at least $7 \times 22 + 1 = 155$ umbrellas.

3. Find the number of integer solutions of $||x| - 2023| < 2020$.

$||x| - 2023| < 2020$ এই সমীকৰণটোৰ কিমানটা অখণ্ড সমাধান থাকিব?

Ans : $||x| - 2023| < 2020 \Rightarrow -2020 < |x| - 2023 < 2020 \Rightarrow 3 < |x| < 4043$. So, $|x|$ can take values $4, 5, 6, \ldots, 4042$ i.e. 4039 values. Thus, x can take $2 \times 4039 = 8078$ values.

4. Real numbers a, b, c satisfy $(2b - a)^2 + (2b - c)^2 = 2(2b^2 - ac)$. Prove that $a + c = 2b$.

a, b, c বাস্তৱ সংখ্যা তিনিটাই $(2b - a)^2 + (2b - c)^2 = 2(2b^2 - ac)$. এই সম্পৰ্কটো সিদ্ধ কৰে। প্ৰমাণ কৰা যে $a + c = 2b$ ।

Ans :

$$\begin{aligned}
&(2b-a)^2 + (2b-c)^2 = 2(2b^2 - ac) \\
\Rightarrow &4b^2 + a^2 - 4ab + 4b^2 + c^2 - 4bc = 4b^2 - 2ac \\
\Rightarrow &(a^2 + c^2 + 2ac) - 4b(a+c) + 4b^2 = 0 \\
\Rightarrow &(a+c)^2 - 2.2b(a+c) + (2b)^2 = 0 \\
\Rightarrow &(a+c-2b)^2 = 0 \\
\Rightarrow &a + c = 2b \\
\Rightarrow &b - a = c - b
\end{aligned}$$

Hence, a, b, c are three consecutive terms in some arithmetic progression.

5. What is the least possible value of $x^2 + y^2 - x - y - xy$ where x, y are real numbers ?

কোনো বাস্তৱ সংখ্যা x, y ৰ বাবে $x^2 + y^2 - x - y - xy$ ৰ সম্ভৱপৰ সৰ্বনিম্ন মান কিমান হ'ব পাৰে?

Ans :

$$\begin{aligned}
&x^2 + y^2 - x - y - xy \\
=&(x^2 + y^2 + 1^2 - x - y - xy) - 1 \\
=&\frac{1}{2}[(x-1)^2 + (y-1)^2 + (x-y)^2] - 1 \\
\geq&\frac{1}{2}(0 + 0 + 0) - 1 = -1
\end{aligned}$$

So, the minimum value is -1.

6. What is the remainder when 128^{2023} is divided by 126?

128^{2023} ক 126 ৰে হৰণ কৰিলে ভাগশেষ কিমান থাকিব?

Ans : $128 = 2^7 \equiv 2 \pmod{126}$. Thus,

$$\begin{aligned} 128^{2023} &\equiv 2^{2023} \pmod{126} \\ &\equiv (2^7)^{289} \pmod{126} \\ &\equiv 2^{289} \pmod{126} \\ &\equiv 2^{7\times 41+2} \pmod{126} \\ &\equiv 2^{41} \times 2^2 \pmod{126} \\ &\equiv 2^{43} \pmod{126} \\ &\equiv 2^{7\times 6+1} \pmod{126} \\ &\equiv 2^6 \times 2^1 \pmod{126} \\ &\equiv 2^7 \pmod{126} \\ &\equiv 2 \pmod{126}. \end{aligned}$$

Required remainder is 2.

7. If $xyz = 1$ find the value of $\left(\frac{1}{1+x+\frac{1}{y}} + \frac{1}{1+y+\frac{1}{z}} + \frac{1}{1+z+\frac{1}{x}}\right)^2$.

যদি $xyz = 1$, তেন্তে $\left(\frac{1}{1+x+\frac{1}{y}} + \frac{1}{1+y+\frac{1}{z}} + \frac{1}{1+z+\frac{1}{x}}\right)^2$ ৰ মান নিৰ্ণয় কৰা।

Ans :

$$\begin{aligned} &\left(\frac{1}{1+x+\frac{1}{y}} + \frac{1}{1+y+\frac{1}{z}} + \frac{1}{1+z+\frac{1}{x}}\right)^2 \\ &= \left(\frac{1}{1+x+xz} + \frac{1}{xyz+y+xy} + \frac{x}{x+xz+1}\right)^2 \\ &= \left(\frac{1}{1+x+xz} + \frac{1}{y(xz+1+x)} + \frac{x}{x+xz+1}\right)^2 \\ &= \left(\frac{1}{1+x+xz} + \frac{xz}{xz+1+x} + \frac{x}{x+xz+1}\right)^2 \\ &= \left(\frac{1+xz+x}{1+x+xz}\right)^2 \\ &= 1 \end{aligned}$$

8. If n is a positive even number, find the last two digits of $(2^{6n}+26)-(6^{2n}-62)$.

কোনো ধনাত্মক যুগ্ম সংখ্যা n ৰ বাবে, $(2^{6n}+26)-(6^{2n}-62)$ ৰ শেষৰ অংক দুটা কি হ'ব নিৰ্ণয় কৰা।

Ans :

$$
\begin{aligned}
&(2^{6n}+26)-(6^{2n}-62)\\
=&(2^{6n}-6^{2n})+88
\end{aligned}
$$

Now, $n = 2m$ (say), as n is even

$$
\begin{aligned}
\therefore 2^{6n}-6^{2n} &= 2^{6\times 2m}-6^{2\times 2m}\\
&= (2^{12})^m-(6^4)^m\\
&= (4096)^m-(1296)^m\\
&\equiv 96^m-96^m (mod 100)\\
&\equiv 0 (mod 100)
\end{aligned}
$$

$\therefore$ Last two digits 88.

9. What is the smallest positive integer having 24 positive divisors?

24 টা ধনাত্মক উৎপাদকবিশিষ্ট আটাইতকৈ সৰু ধনাত্মক অখণ্ড সংখ্যাটো কি?

Ans : The number of positive divisors of a number of the form $p^a\times q^b\times r^c\times s^d$ where p,q,r,s are distinct primes is $(a+1)(b+1)(c+1)(d+1)$. Since $24 = 12\times 2 = 6\times 4 = 4\times 3\times 2 = 3\times 2\times 2\times 2$, so numbers of the form p^{23} or $p^{11}\times q$ or $p^5\times q^3$ or $p^3\times q^2\times r$ or $p^2\times q\times r\times s$ have 24 divisors. To find the smallest number having 24 positive divisors, we need to consider prime factors as small as possible, raised to small powers. Thus, the possible choices are 2^{23}, $2^{11}\times 3$, $2^5\times 3^3$, $2^3\times 3^2\times 5$ and $2^2\times 3\times 5\times 7$. Out of these, the smallest is $2^3\times 3^2\times 5 = 360$.

10. If $a,b,c\neq 0$, prove that $\dfrac{a^2+b^2}{c^2}+\dfrac{b^2+c^2}{a^2}+\dfrac{c^2+a^2}{b^2}\geq 6$.

যদি $a,b,c\neq 0$, প্ৰমাণ কৰা যে $\dfrac{a^2+b^2}{c^2}+\dfrac{b^2+c^2}{a^2}+\dfrac{c^2+a^2}{b^2}\geq 6$.

Ans : Since $a,b,c\neq 0$, so $a^2,b^2,c^2>0$.

$$
\begin{aligned}
&\frac{a^2+b^2}{c^2}+\frac{b^2+c^2}{a^2}+\frac{c^2+a^2}{b^2}\\
=&\frac{a^2}{c^2}+\frac{b^2}{c^2}+\frac{b^2}{a^2}+\frac{c^2}{a^2}+\frac{c^2}{b^2}+\frac{a^2}{b^2}\\
\geq&6\left(\frac{a^2}{c^2}\times\frac{b^2}{c^2}\times\frac{b^2}{a^2}\times\frac{c^2}{a^2}\times\frac{c^2}{b^2}\times\frac{a^2}{b^2}\right)^{1/6}=6
\end{aligned}
$$

using AM-GM inequality for the 6 quantities $\frac{a^2}{c^2},\frac{b^2}{c^2},\frac{b^2}{a^2},\frac{c^2}{a^2},\frac{c^2}{b^2},\frac{a^2}{b^2}$.

11. Let $p(x)$ be a polynomial of degree 10 with non-negative integer coefficients. The remainder when $p(x)$ is divided by $(x-1)$ is 3. How many such polynomials are there ?

ধৰা হ'ল $p(x)$ এটা 10 মাত্ৰাৰ বহুপদ ৰাশি যাৰ সহগসমূহ হৈছে অঋণাত্মক অখণ্ড সংখ্যা। $p(x)$ ক $(x-1)$ ৰে হৰণ কৰিলে ভাগশেষ 3 থাকে। তেনেকুৱা কিমানটা বহুপদ ৰাশি থাকিব?

Ans :

$$P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{10}x^{10}, \qquad a_{10} \neq 0$$

According to question,

$$P(1) = 3$$
$$\Rightarrow a_0 + a_1 + a_2 + \cdots + a_{10} = 3$$

Since $a_{10} \neq 0$, so let $a_{10} = 1 + y$, $y \geq 0$

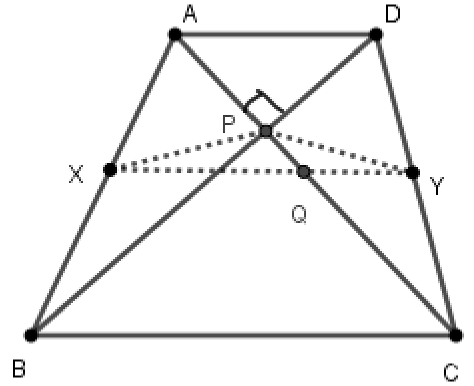
$$\therefore a_0 + a_1 + a_2 + \cdots + a_9 + (1+y) = 3$$
$$\Rightarrow a_0 + a_1 + a_2 + \cdots + a_9 + y = 2$$

Number of non-negative integer solutions of the equation is $\dfrac{(2+10)!}{2!10!} = \dfrac{12 \times 11}{2} = 66$. Thus, there are 66 such polynomials.

12. In quadrilateral $ABCD$, $AD \parallel BC$, diagonals AC and BD are perpendicular to each other, X and Y are mid-points of AB and CD respectively. Prove that $AB + CD \geq AD + BC$.

$ABCD$ চতুৰ্ভুজটোত $AD \parallel BC$, AC আৰু BD কৰ্ণদুডাল পৰস্পৰ লম্ব, X আৰু Y হৈছে যথাক্ৰমে AB আৰু CDৰ মধ্যবিন্দু। প্ৰমাণ কৰা যে $AB + CD \geq AD + BC$.

Ans :



Since ΔAPB is right-angled triangle and X is mid point of AB. So, $PX = AX = XB = \frac{1}{2}AB$. Similarly, DPC is the right-angled triangle, so $PY = DY = YC = \frac{1}{2}DC$. In ΔPXY,

$$\begin{aligned}
&PX + PY \geq XY \\
\Rightarrow &\frac{1}{2}AB + \frac{1}{2}DC \geq XQ + QY \\
\Rightarrow &\frac{1}{2}AB + \frac{1}{2}DC \geq \frac{1}{2}BC + \frac{1}{2}AD \\
\Rightarrow &AB + CD \geq AD + BC
\end{aligned}$$

13. Let $S(r)$ denote the sum of the infinite geometric series $17 + 17r + 17r^2 + 17r^3 + \ldots$ for $-1 < r < 1$. If $S(a) \times S(-a) = 2023$, find $S(a) + S(-a)$.

    ধৰা হ'ল কোনো $-1 < r < 1$ ৰ বাবে, $S(r)$ য়ে $17 + 17r + 17r^2 + 17r^3 + \ldots$ এই গুণোত্তৰ শ্ৰেণীটোৰ যোগফলক সূচায়। যদি $S(a) \times S(-a) = 2023$, তেন্তে $S(a) + S(-a)$ ৰ মান নিৰ্ণয় কৰা

    Ans : $S(a) = \dfrac{17}{1-a}$, $S(-a) = \dfrac{17}{1+a}$

$$\begin{aligned}
&S(a) \times S(-a) = 2023 \\
\Rightarrow &\frac{17}{1-a} \times \frac{17}{1+a} = 2023 \\
\Rightarrow &\frac{1}{(1-a)(1+a)} = 7 \\
\therefore &S(a) + S(-a) = \frac{17}{1-a} + \frac{17}{1+a} = \frac{34}{(1-a)(1+a)} = 34 \times 7 = 238
\end{aligned}$$

14. Find all possible triples of integers a, b, c satisfying $a+b-c = 1$ and $a^2+b^2-c^2 = -1$.

    $a + b - c = 1$ আৰু $a^2 + b^2 - c^2 = -1$ এই সমীকৰণ দুটা সিদ্ধ কৰা আটাইবোৰ সম্ভৱপৰ অখণ্ড সমাধান a, b, c নিৰ্ণয় কৰা।

    Ans :

$$a + b - c = 1 \Rightarrow a + b = c + 1 \tag{1}$$

$$\begin{aligned}
&a^2 + b^2 - c^2 = -1 \\
\Rightarrow &(a+b)^2 - 2ab = c^2 - 1 \\
\Rightarrow &(c+1)^2 - 2ab = c^2 - 1 \\
\Rightarrow &c^2 + 2c + 1 - 2ab = c^2 - 1 \\
\Rightarrow &2ab = 2c + 2 \\
\Rightarrow &ab = c + 1 && (2)
\end{aligned}$$

From 1 and 2, we get

$$a + b = ab \tag{3}$$

If $a = 0$, then $b = 0$. Also, if $b = 0$ then $a = 0$. Thus, $a = 0, b = 0$ is a solution of 3. In this case, $c = -1$ (from 1). So, $a = b = 0, c = -1$ is a solution.
Now, other solutions, if any, of 3 will require both a and b to be non-zero. Using 3,

$$\begin{aligned} &a(1-b) = -b \\ \Rightarrow &a(b-1) = b \\ \because\ &a \neq 0 \textit{ and } a \textit{ and } b \textit{ are integers, so } a|b \end{aligned} \tag{4}$$

Again, from 3

$$\begin{aligned} &b(a-1) = a \\ &\because a \neq 0 \textit{ and } a \textit{ and } b \textit{ are integers, so } b|a \end{aligned} \tag{5}$$

From 4 and 5, $a = b$. Thus, using 3

$$a + a = a^2 \tag{6}$$

$$\Rightarrow a^2 = 2a \tag{7}$$

$$\Rightarrow a = 2 (\because a \neq 0) \tag{8}$$

$\therefore b = 2$ and $c = 2 + 2 - 1 = 3$.

15. Let $f(x)$ be a polynomial of degree 3 with real coefficients satisfying $|f(x)| = 12$ for $x = 1, 2, 3, 5, 6, 7$. Find $|f(0)|$. (Typo corrected)

ধৰা হ'ল $f(x)$ হৈছে বাস্তৱ সহগবিশিষ্ট 3 মাত্ৰাৰ এটা বহুপদ যাতে $x = 1, 2, 3, 5, 6, 7$ ৰ বাবে $|f(x)| = 12$। তেন্তে $|f(0)|$ ৰ মান নিৰ্ণয় কৰা।

Ans : (There was a typo in the original question. The values of x were given to be $x = 1, 2, 3, 4, 5, 6, 7$. Thus, $f(x)$ is either 12 or -12 at $x = 1, 2, 3, 4, 5, 6, 7$. So, by Pigeonhole Principle, f either takes the value 12 at least at 4 different x or -12 at least at four different x. But that is not possible as f is a polynomial of degree three. Students who have provided similar justifications have been awarded marks.)

Without loss of generality, let $f(1) = 12$ (If $f(1) = -12$ then take $-f(x)$ as the polynomial, which leaves $|f(0)|$ unchanged.) Because f is third-degree, write

$$f(x) - 12 = a(x-1)(x-b)(x-c) \tag{9}$$

$$f(x) + 12 = a(x-d)(x-e)(x-f) \tag{10}$$

where, b, c, d, e, f clearly must be a permutation of $2, 3, 5, 6, 7$ from the given condition. Thus $b + c + d + e + f = 2 + 3 + 5 + 6 + 7 = 23$.
Subtracting the two equations 9, 10 we get,

$$-24 = a[(x-1)(x-b)(x-c) - (x-d)(x-e)(x-f)]$$

Comparing the coefficients of x^2, we get $1+b+c = d+e+f$ and thus both values equal to $\frac{24}{2} = 12$. As a result, $\{b, c\} = \{5, 6\}$. As a result, $-24 = a(12)$ and so $a = -2$. Now, we easily deduce that $f(0) = (-2) \cdot (-1) \cdot (-5) \cdot (-6) + 12 = 72$, and so removing the without loss of generality gives $|f(0)| = 72$.

16. n is a positive integer such that the product of all its positive divisors is n^3. Find all such n less than 100.

n এটা ধনাত্মক অখণ্ড সংখ্যা যাতে ইয়াৰ আটাইবোৰ ধনাত্মক উৎপাদকৰ পূৰণফল n^3 হয়। 100 তকৈ সৰু তেনেকুৱা আটাইবোৰ n বিচাৰি উলিওৱা।

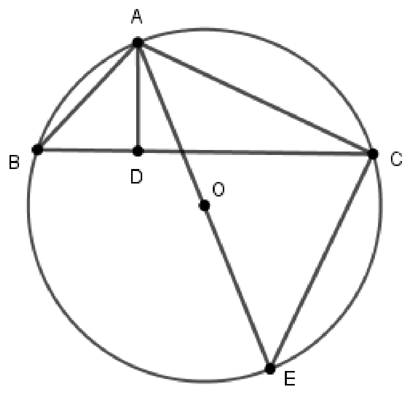
Ans : Let $d_1, d_2, \ldots, d_k$ be the positive divisors of n in increasing order. By question $d_1 d_2 \ldots d_k = n^3$. Observe that $d_1 d_k = n$, $d_2 d_{k-1} = n$, $d_3 d_{k-2} = n$ and so on. Thus, regrouping the divisors in $d_1 d_2 \ldots d_k = n^3$, we get $(d_1 d_k)(d_2 d_{k-1})(d_3 d_{k-2}) \ldots = n^3$. But since each product in the parentheses in LHS is n, so that means there should be only six divisors. Thus, n should be of the form p^5 or pq^2 where p, q are distinct primes. The only n less than 100 of the form p^5 is $2^5 = 32$. The numbers of the form pq^2 are $2 \times 3^2 = 18$, $3 \times 2^2 = 12$, $2 \times 5^2 = 50$, $5 \times 2^2 = 20$, $2 \times 7^2 = 98$, $7 \times 2^2 = 28$, $11 \times 2^2 = 44$, $13 \times 2^2 = 52$, $17 \times 2^2 = 68$, $19 \times 2^2 = 76$, $23 \times 2^2 = 92$, $3 \times 5^2 = 75$, $5 \times 3^2 = 45$, $7 \times 3^2 = 63$, $11 \times 3^2 = 99$.

17. If in $\triangle ABC$, AD is the altitude and AE is the diameter of the circumcircle through A, then prove that $AB \cdot AC = AD \cdot AE$. Use this result to show that if $ABCD$ is a cyclic quadrilateral then show that

$AC \cdot (AB \cdot BC + CD \cdot DA) = BD \cdot (DA \cdot AB + BC \cdot CD)$.
$\triangle ABC$ ত AD উন্নতি আৰু AE হৈছে A ৰ মাজেৰে পাৰ হৈ যোৱা পৰিবৃত্তটোৰ ব্যাস। তেন্তে প্ৰমাণ কৰা যে $AB \cdot AC = AD \cdot AE$. এই ফলাফলটো ব্যৱহাৰ কৰি দেখুওৱা যে $ABCD$ এটা চক্ৰীয় চতুৰ্ভুজ, তাৰ পাছত দেখুওৱা যে

$AC \cdot (AB \cdot BC + CD \cdot DA) = BD \cdot (DA \cdot AB + BC \cdot CD)$.

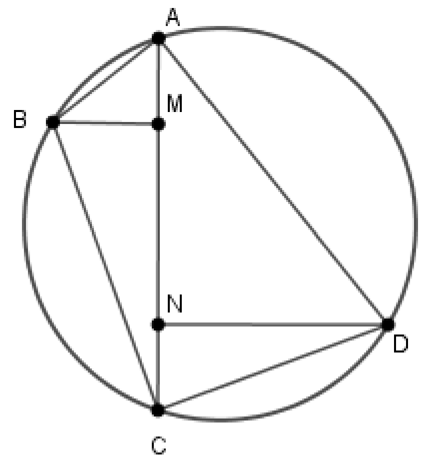
Ans :



In ΔABD *and* ΔAEC,

$$\angle ADB = \angle ACE (= 90^\circ)$$
$$\angle ABD = \angle CEA$$

$\therefore \Delta ABD \sim \Delta AEC$ which implies $\dfrac{AB}{AE} = \dfrac{AD}{AC} \Rightarrow AB.AC = AD.AE.$



Now consider a cyclic quadrilateral ABCD in a circle of radius R. Using the above result, we have

$$AB.BC = 2R.BM$$
$$DA.DC = 2R.DN$$

Hence,

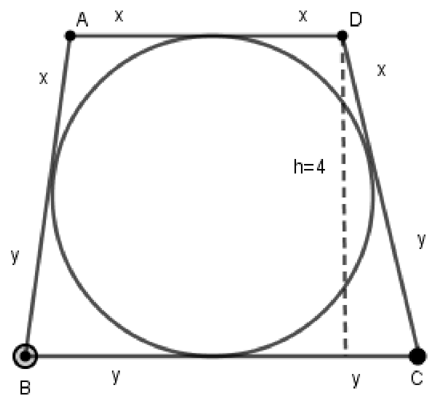
$$AC.[AB.BC + CD.DA]$$
$$= AC.2R.BM + AC.2R.DN$$
$$= 2R[2\Delta ABC + 2\Delta ACD]$$
$$= 4RS$$

where $S =$ the area of the quadrilateral $ABCD$.
Similarly, $BD.[DA.AB + BC.CD] = 4RS$, and then result follows.

18. A circle of radius 2 is inscribed in an isosceles trapezoid with the area of 28. Find the length of the side of the trapezoid.

28 বৰ্গ একক কালিৰ সমদ্বিবাহু ট্ৰেপিজিয়াম ক্ষেত্ৰ এটাত 2 একক ব্যাসাৰ্ধৰ বৃত্ত এটা অন্তৰ্বিষ্ট কৰা হৈছে। ট্ৰেপিজিয়ামটোৰ বাহুৰ দৈৰ্ঘ্য নিৰ্ণয় কৰা।

Ans :



The height and area of the trapezoid are 4 and 28 respectively. Thus,

$$\frac{1}{2} \times 4 \times (2x + 2y) = 28$$
$$\Rightarrow x + y = 7 \qquad (11)$$

Also,

$$\left(\frac{2y - 2x}{2}\right)^2 + 4^2 = (x + y)^2$$
$$\Rightarrow 4xy = 4 \times 4$$
$$\Rightarrow xy = 4 \qquad (12)$$

From 11 and 12, we have,

$$x + \frac{4}{x} = 7$$
$$\Rightarrow x^2 - 7x + 4 = 0$$
$$\Rightarrow x = \frac{7 \pm \sqrt{49 - 16}}{2}$$

So, $2x = 7 - \sqrt{33}, 2y = 7 + \sqrt{33}$ (As we assumed $x < y$). So, the sides of the trapezoid are $2x = 7 - \sqrt{33}, x + y = 7, 2y = 7 + \sqrt{33}$ and $x + y = 7$.