Assam Academy of Mathematics

MATHEMATICS OLYMPIAD

September 9, 2018 (Sunday)

Category-II ***(for Classes VII & VIII)***

(Figures in the margin indicate full marks for the questions.)

Q. 1. 16

(a) Three boys A, B, C were asked to divide a certain number by 1001 by the method of factors. They took the factors in the order 13, 11, 7; 7, 11, 13 and 11, 7, 13 respectively. If the first boy A obtained 3, 2, 1 as successive remainders, find the successive remainders obtained by other two boys B and C.

Soln. We have 1001 = 13 × 11 × 7.

If N be the number to be divided by 13, 11, 7 leaving remainders 3, 2, 1 then

N = $13\times p+3$ where $p=11\times q+2$ and $q=7\times r+1$

$$=13\left(11\times q+2\right)+3$$

$$=13\times 11\times\left(7\times r+1\right)+26+3$$

$$=13\times 11\times 7\times r+143+29$$

$$=13\times 11\times 7\times r+172$$

Now,

$$\begin{array}{r|l}
7 & 172 \\ \hline
11 & 24 \;—\; 4 \\ \hline
13 & 2 \;—\; 2 \\ \hline
 & 0 \;—\; 2
\end{array}$$

∴ Successive remainders obtained by B are 4, 2, 2

Also

$$\begin{array}{r|l}
11 & 172 \\ \hline
7 & 15 \;—\; 7 \\ \hline
13 & 2 \;—\; 1 \\ \hline
 & 0 \;—\; 2
\end{array}$$

∴ Successive remainders obtained by C are 7, 1, 2.

(b) A three digit number 4a3 is added to another three digit number 984 to give a four digit number 13b7, which is divisible by 11. Find the value of 2a+b.

Soln. We have

$$\begin{array}{r} 984 \\ 4a3 \\ \hline 13b7 \end{array}$$

Where $13b7$ is divisible by 11.

This means that $1-3+b-7=b-9$ is 0 or divisible by 11

i.e., $b=9$ or $b-9=11, 22, 33,...$

Since b is a digit, only possible value of b is 9.

Now, $8+a=9 \Rightarrow a=1.$

Therefore $2a+b=2\times1+9=11$

Q. 2. 10

(a) 8 taps are fitted to a water tank. Some of them are water taps to fill the tank and the remaining are outlet taps used to empty the tank. Each water tap can fill the tank in 12 hours and each outlet tap can empty it in 36 hours. On opening all the taps, if the tank is filled in 3hours find the number of water taps.

Soln. Let x be the number of water taps. Then 8 – x is the number of outlet taps. In 1 hour each water tap can fill $\frac{1}{12}$ of the whole tank. Hence x water taps can fill $\frac{x}{12}$ part of the whole tank in 1 hour.

Also an outlet tap can empty $\frac{1}{36}$ part of the tank in 1 hour. Hence 8 – x outlet taps can empty $\frac{8-x}{36}$ parts of the tank in 1 hour.

Thus, in 1 hour $\frac{x}{12}-\left(\frac{8-x}{36}\right)$ part of the tank can be filled up in 1 hour.

In other words, to fill $\frac{3x-8+x}{36}=\frac{4x-8}{36}=\frac{x-2}{9}$ part of the tank, time required = 1 hour.

Hence time required to fill the whole tank is $\frac{9}{x-2}$ hours.

By condition, $\frac{9}{x-2}=3$

$\Rightarrow x-2=3$

$\Rightarrow x=5$

Number of water taps is 5.

(b) Prove that $2^{567} > (30)^{100}$ 6

Soln. $2^{567} > 2^{500}$

$=\left(2^5\right)^{100}$

$=(32)^{100}$

$> 30^{100}$

Q. 3.

(a) Find quotient and remainder when the number consisting of 1001 sevens is divided by the number 1001. 5

Soln. 777777 is exactly divisible by 1001 with quotient 777. $\underbrace{777............7}_{1001\text{ digit}}$ when divided by 1001 will give a quotient $\underbrace{|777000/777000/............/777000/}_{166\text{ group}}$

with $\frac{77777}{1001}$ remaining.

But $\frac{77777}{1001}=77+\frac{700}{1001}$

Hence, required quotient = $\underbrace{\underbrace{777000}\underbrace{777000}....\underbrace{777000}}_{166\text{ times}}\ 77$

and remainder = 700

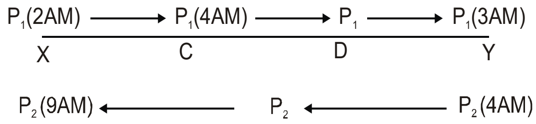
(b) On a particular day a person starts walking from a place X at 2 A.M. and reaches place Y at 5 A.M. A second person starts walking from place Y at 4 A.M. and reaches place X at 9 A.M. on the same day. At what time do they cross each other. 5

Soln. Let us denote the persons starting from X and Y by P_1 and P_2 respectively.

Also let XY= a km.

Let P_1 move to C when P_2 starts moving 4 am. Then let them meet at D such that XD = x

Speed of P_1 is $\frac{XY}{3} = \frac{a}{3}$ km/hour.



Speed of P_2 is $\frac{XY}{5} = \frac{a}{5}$ km/hour.

Then $XC = \frac{a}{3} \times 2$

Now time taken to cover CD by P_1 is same as the time taken by P_2 to cover DY.

$$\therefore \frac{CD}{\frac{a}{3}} = \frac{DY}{\frac{a}{5}}$$

i.e. $\frac{3}{a}\left(x - \frac{2a}{3}\right) = \frac{5}{a}\left(a - x\right)$

$$\Rightarrow \frac{3x}{a} + \frac{5x}{a} = 5 + 2 = 7$$

$$\Rightarrow \frac{8x}{a} = 7$$

$$\Rightarrow x = \frac{7a}{8}$$

The two persons meet at a distance of $\frac{7a}{8}$ from X.

(c) A square sheet of paper ABCD is so folded that B falls on the midpoint M of CD. Prove that the crease will divide BC in the ratio 3:5. 6

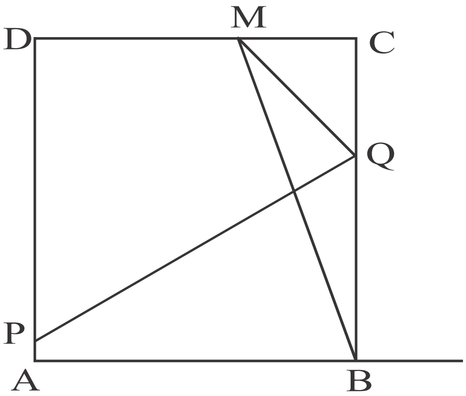
Soln. Let PQ denote the crease formed by the fold of the paper so that B falls at the mid point M of CD.

Clearly the crease will be a perpendicular bisector of BM.

Hence QM = QB.

Let a be the side of the square and QC = x.

The $\mathrm{CM} = \frac{a}{2},\ \mathrm{BQ} = a - x = \mathrm{MQ}$



From $\Delta\mathrm{CMQ},\ \mathrm{MQ}^2 = \mathrm{CQ}^2 + \mathrm{CM}^2$

$$\Rightarrow \mathrm{BQ}^2 = \mathrm{CQ}^2 + \mathrm{CM}^2$$

$$\Rightarrow (a-x)^2 = x^2 + \left(\frac{a}{2}\right)^2$$

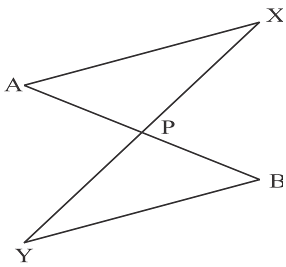
$$\Rightarrow x = \frac{3a}{8} \text{ and } \mathrm{BQ} = \frac{5a}{8}$$

$\therefore\ \mathrm{BQ} : \mathrm{QC} = 5 : 3$

Q. 4.

(a) AB is a line segment. AX and BY are two equal line segments drawn along opposite directions of line AB such that $\mathrm{AX} \parallel \mathrm{BY}$. If line segments AB and XY intersect each other at point P, prove that each of the line segments is bisected at P. 6

Soln. Clearly $\Delta\mathrm{AXP}$ and $\Delta\mathrm{BYP}$ are equiangular and therefore they are similar. This means,



$$\frac{\mathrm{AX}}{\mathrm{BY}} = \frac{\mathrm{AP}}{\mathrm{PB}} = \frac{\mathrm{XP}}{\mathrm{YP}}$$

But AX=BY

Therefore, $\mathrm{AP} = \mathrm{PB},\ \mathrm{XP} = \mathrm{YP}$.

In other words AB and XY are bisected at P.

(b) The minute hand of a clock is 10 cm. long. Find the area of the face of the clock described by the minute hand between 9 A.M. and 9.35 A.M. 5

Soln. Assuming the face of the clock to be circular with 10cm as radius, its whole area is

$$\pi(10cm)^2 = 100\pi cm^2$$

The perimeter having 60 divisions, for a diffeence of 1 minute the area described will be

$$\frac{100\pi}{60} cm^2$$

Hence the area described between 9 AM and 9.35 AM is $35 \times \frac{100}{60} \pi cm^2$

$$= \frac{175}{3} \pi cm^2$$

(c) Water flows at the rate of 10 meters per minute through a cylindrical pipe whose internal radius is 0.5 cm. How long would it take to fill a conical vessel whose radius at the top is 20 cm. and depth is 21 cm? 5

Soln. Amount of water moving per minute is $\pi(0.5)^2 \times 10{,}00\, cm^3$

Volume of the cone is $\frac{1}{3}\pi(20)^2 \times 21 cm^3$

Hence time required to fill the vessel is $\frac{\frac{1}{3}\pi(20)^2 \times 21}{\pi(0.5)^2 \times 10{,}00}$ minutes

$$= \frac{20 \times 20 \times 21}{3 \times 0.25 \times 10{,}00} \text{ minutes}$$

$$= \frac{400 \times 21}{250 \times 3} = \frac{280\not{0}}{25\not{0}} = 11\frac{1}{5} \text{ minutes}$$

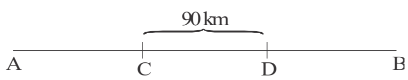
= 11 minute 12 seconds.

Q. 5.

(a) A train, an hour after starting meets with an accident which detains it for 30 minutes. After this the train proceeds at $\frac{3}{4}$ of its former speed and arrives $3\frac{1}{2}$ hours late. Had the accident happened 90 K.M. farther along the line, it would have arrived only 3 hours late. Find the length of the Journey. 8

Soln. Let AB = l be the total length of the journey and V the velocity of the train. Thus the train travelled V km in 1 hour.
i.e., AC = V km.

Actual time of journey from A to B is $\frac{l}{V}$



By condition,

$$1+\frac{1}{2}+\frac{l-\text{V}}{\frac{3}{4}\text{V}}=\frac{l}{\text{V}}+3½$$

$$\Rightarrow \frac{4(l-\text{V})}{3\text{V}}-\frac{l}{\text{V}}=2$$

$$\Rightarrow \frac{4l}{3\text{V}}-\frac{l}{\text{V}}-\frac{4}{3}=2$$

$$\Rightarrow \frac{l}{3\text{V}}=\frac{10}{3}$$

$$\Rightarrow l=10\text{V} \qquad \text{— (i)}$$

Had the accident occurred 90 km farther, we have

$$1+\frac{90}{\text{V}}+\frac{1}{2}+\frac{l-\text{V}90}{\frac{3}{4}\text{V}}=\frac{l}{\text{V}}+3$$

$$\Rightarrow \frac{90}{\text{V}}+\frac{4}{3\text{V}}(l-\text{V}-90)-\frac{l}{\text{V}}=\frac{3}{2}$$

$$\Rightarrow \frac{270+4l-4\text{V}-360-3l}{3\text{V}}=\frac{3}{2}$$

$$\Rightarrow l-4\text{V}-90=\frac{9\text{V}}{2}$$

$$\Rightarrow 2l-8\text{V}-9\text{V}=180$$

$$\Rightarrow 20\text{V}-17\text{V}=180 \qquad \text{[using (i)]}$$

$$\Rightarrow \text{V}=60$$

Thus l = 10V = 10 × 60 = 600 km.

(b) Find the least number which on being divided by 5, 6, 8, 9, 12 leaves in each case a remainder 1 but when divided by 13 leaves no remainder. 7

Soln. LCM of 5, 6, 8, 9, 12 is 360

The required least number must be 1 more than a multiple of 360 so that it is divisible by 13.

By inspection, it is found to be equal to $360\times10+1=3601$

(c) Find the units digit of the number $(4444)^{4444}$ 5

Sonl. Unit's digit of the given number is the same as unit's digit of 4^{4444}

Now,

Unit's digit of 4^1 — 4

Unit's digit of 4^2 — 6

Unit's digit of 4^3 — 4

Unit's digit of 4^4 — 6

Thus units of digit of 4^n is 4 or 6 according as *n* is odd or even.

4444 being even, unit's digit of 4^{4444} and therefore of $(4444)^{4444}$ is 6.

Q. 6.

(a) If (4x – y) is a multiple of 3, show that $4x^2 + 7xy - 2y^2$ is divisible by 9. 6

Soln. $4x^2+7xy-2y^2$

$=4x^2+8xy-xy-2y^2$

$=4x(x+2y)-y(x+2y)$

$=(4x-y)(x+2y)$ — (i)

Now $4x-y=3k$, (k an integer) by condition

i.e., $4x-3k=y$.

Therefore $x+2y=x+8x-6k=9x-6k=3(3x-2k)$

Thus, $4x^2+7xy-2y^2=(4x-y)(x+2y)$ by (i)

$=3k\times3(3x-2k)$

$=9k(3x-2k)$

Which is a multiple of 9.

(b) Find the sum of all the digits of the results of the subtraction $10^{99} - 99$.
4

Soln. $10^{99}-99=\underbrace{1000.....00}_{99\text{ zeors}}-99$

$=\underbrace{999.........9}_{97\text{ nines}}01$

Hence the sum of the digits of $10^{99} - 99$ is

$$9 \times 97 + 1 = 874$$

(c) In a car, number plate contains only 3 or 4 digits not containing the digit 0. What is the maximum number of cars that can be numbered? 6

Soln. When the number plate contains three digits excluding 0, each digit can be filled in 9 ways. Therefore, the number of plate that can be made is 9^3.

Similarly when the plate contains 4 digit numbers, the total number of plates that can be made is 9^4.

Therefore, total number of cars for which 3 digit or 4 digit number plates are used is

$$9^3 + 9^4 = 9^3(1+9) = 7290$$