Assam Academy of Mathematics

SOLVED PAPER MATHLETICS, 2017

(Classes IX, X and X appeared)

Time : 3 Hours (11am to 2pm.) Marks : 100

1. Prove that the sum of the cubes of the legs of a right angled triangle is less than the cube of the hypotenuse.

Ans. Let a, b, c denote the sides BC, CA and AB respectively of a rght angled triangle ABC with right angle at A

Then by Pythagorus theorem

$BC^2=CA^2+AB^2$

i.e, $a^2=b^2+c^2$

Then $b<a,\ c<a$

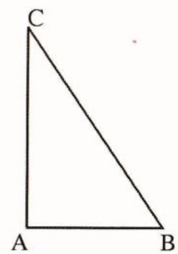
Now $a^3=a(a^2)$

$=a(b^2+c^2)$

$=ab^2+ac^2$

$>bb^2+cc^2$

$=b^3+c^3$



Thus $b^3+c^3<a^3$

Hence Proved

2. Find the integral solutions of the following equation.

$2x^2y^2+y^2-6x^2-12=0$

Ans. Rewriting the equation we have

$(2y^2-6)x^2=12-y^2$

$\therefore\ x^2=\dfrac{12-y^2}{2y^2-6}=\dfrac{12-y^2}{2(y^2-3)}$

Now, x^2 being positive, we must have

$3<y^2<12$

Clearly $y^2=4$ and $y^2=9$

i.e. $y=\pm 2$ and $y=\pm 3$

For $y=\pm 2,\ x^2=4$

i.e. $x=\pm 2$

For $y=\pm 3,\ x^2=\dfrac{3}{12}=\dfrac{1}{4}$, which is non integr

Hence $y=\pm 3$ is impossible

Thus the possible values of (x, y) are

(2, 2), (2, -2), (-2, 2), (-2, -2)

3. Solve the following inequation.

$|x-1-x^2|\le|x^2-3x+4|$

Ans. Given inequation,

$|x-1-x^2|\le|x^2-3x+4|$

Squaring both sides–

$(x-1-x^2)^2\le(x^2-3x+4)^2$

or $(x^2-3x+4)^2-(x-1-x^2)^2\ge 0$

or $(x^2-3x+4+x-1-x^2)(x^2-3x+4-x+1+x^2)\ge 0$

or $(-2x+3)(2x^2-4x+5)\ge 0$

or $(2x-3)\left(2(x-1)^2+3\right)\leq 0$

or $2x-3\leq 0$, since $2(x-1)^2+3>0$ for any x

i.e. $x\leq\frac{3}{2}$

<u>Another method for Q.3</u>

$$|x-1-x^2|\leq|x^2-3x+4|$$

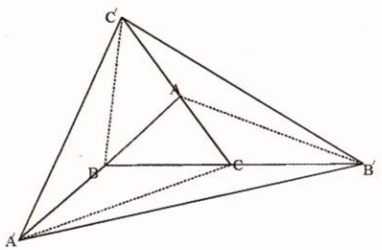
i.e. $|x^2-x+1|\leq|x^2-3x+4|$

i.e. $\left|\left(x-\frac{1}{2}\right)^2+\frac{3}{4}\right|\leq\left|\left(x-\frac{3}{2}\right)^2+\frac{7}{4}\right|$

which clearly shows that the above is true for all $x\leq\frac{3}{2}$

4. In a triangle ABC, A is reflected at B to A′, B is reflected at C to B′ and C is reflected at A to C′. Find $|A'B'C'|$ in terms of $|ABC|$ where $|A'B'C'|$ and $|ABC|$ respectively denote the areas of the triangles A′B′C′ and ABC.

Ans. AB′, BC′ and CA′ are joined by segments



Clearly, $|A'BC|=|A'CB'|$ and $|ABC|=|ACB'|$

$\therefore$ $|B'C'C|=2|ABC|$

similarly $|C'A'A|=2|ABC|$ and $|A'B'B|=2|ABC|$

$\therefore$ $|A'B'C'|=|A'BB'|+|B'CC'|+|C'AA'|+|ABC|$

$=2|ABC|+2|ABC|+2|ABC|+|ABC|$

$=7|ABC|$

5. The polynomial

$ax^4+bx^3+cx^2+dx+e$

with integer coefficients is divisible by 7 for every integer x, show that $7|a, 7|b, 7|c, 7|d$ and $7|e$

Ans. Taking $x=0, 1, -1, 2, -2$, we have

$7|e$ ... ... (i)

$7|a+b+c+d+e$ ... ... (ii)

$7|a-b+c-d+e$ ... ... (iii)

$7|16a+8b+4c+2d+e$ ... ... (iv)

and $7|16a-8b+4c-2d+e$ ... ... (v)

Adding (ii) and (iii) $7|2(a+c+e)$

$\Rightarrow 7|(a+c+e)$

$\Rightarrow 7|a+c$ since $7|e$ ... ... (vi)

Subtracting (iii) from (ii) $7|2(b+d)$

i.e. $7|b+d$

Adding (iv) and (v) $7|32a+8c+2e$

i.e. $7|16a+4c+e$

$\Rightarrow 7|16a+4c$

$\Rightarrow 7|4a+c$ ... ... (vii)

(vii) – (vi) $\Rightarrow 7|3a$

$\Rightarrow 7|a$

From (vi), therfore $\Rightarrow 7|c$

Again, (iv) – (v) $\Rightarrow 7|16b+4d$

$\Rightarrow 7|4b+d$

Now, $7|b+d$ and $7|4b+d \Rightarrow 7|3b$

$\Rightarrow 7|b$ and so $\Rightarrow 7|d \quad \because \quad 7|b+d$

This $7|a, 7|b, 7|c, 7|d$ and $7|e$

6. Show that the sum of squares of five successive positive integers is not a square.

Ans. Let us consider the five (consecutive) positive integers as,

$n-2,\ n-1,\ n,\ n+1,\ n+2$

Sum of squaares of these numbers is

$(n-2)^2+(n-1)^2+n^{2+}(n+1)^2+(n+2)^2$

$=n^2+2(n^2+1^2)+2(n^2+2^2)$

$=5n^2+10$

$=5(n^2+2)$

If this number is a perfect square than n^2+2 must have a factor equal to 5, since 5 is already occurring as a factor

i.e. 5 must divide n^2+2

i.e. $n^2+2 = 5q$ for some integer q

i.e. $n^2 = 5q-2$, q being an integer

But the square of an integer can be of the from 4k or 4k+1 only and not of the form 5q–2

Hence the resulting number is not a perfect square.

7. Do there exist positive integers x, y such that $x+y$, $2x+y$ and $x+2y$ are all perfect squares? Justify.

Ans. If possible, let there exist intergers x and y such that

$x+y, 2x+y$ and $x+2y$ are all perfect squares.

Then

$x+y = a^2$, $2x+y = b^2$ and $x+2y = c^2$ for some integers a, b and c.

Now

$b^2+c^2 = 3x+3y = 3(x+y) = 3a^2$

Thus

$3 | b^2+c^2$

But, by Division Algorithm, any integer can be expressed in the form 3k, 3k+1 or 3k+2

Hence the square of any integer leaves remainder 0 or 1 upon division by 3

Therefore $b^2 \equiv 0$ or 1 (mod 3)

and $c^2 \equiv 0$ or 1 (mod 3)

But if $b^2 \equiv 1 \pmod 3$ and $c^2 \equiv 1 \pmod 3$

then we have $b^2+c^2 \equiv 2 \pmod 3$

which is contradictory to the result $3 | b^2+c^2$

Hence, $b^2 \not\equiv 1 \pmod 3$ and $c^2 \not\equiv 1 \pmod 3$

i.e. $b^2 \equiv 0 \pmod 3$ and $c^2 \equiv 0 \pmod 3$

i.e. $3 | b^2$ and $3 | c^2$

$\Rightarrow$ $3 | b$ and $3 | c$

$\Rightarrow$ $b = 3r$ and $c = 3s$ for some integers r and s

Now $3a^2 = b^2+c^2$

$= 9r^2+9s^2$

$= 9(r^2+s^2)$

i.e. $a^2 = 3(r^2+s^2)$

$\Rightarrow$ $3 | a^2$ and therefore $3 | a$ also.

Thus if (a, b, c) satisfly the relation

$b^2+c^2 = 3a^2$

then $\left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right)$ also satisfy it

The process can be continued infinitely. Since a, b, c are finite positive integers, we will arrive ultimately at the trivial solution (0, 0, 0) for $3a^2 = b^2+c^2$

Hence there exist no positive integers a, b, c satisfying the three eqnations.

i.e. There are no integral values of x any y for which

$x+y$, $2x+y$ and $x+2y$ can be perfact squares.

8. If $a_i \in \{-1, 1\}, i = 1, 2, 3, \ldots, n$ and $a_1a_2+a_2a_3+a_3a_4+\ldots+a_na_1=0$ then show that $4 | n$.

Ans. There are n terms in $a_1a_2+a_2a_3+\ldots\ldots\ldots+a_na_1$ and each term is either 1 or −1

Since $a_1a_2+a_2a_3+\ldots\ldots\ldots+a_na_1 = 0$, there must be even number of terms, half of which are 1's and other half are −1's.

Thus n = 2k for some integer k.

But $a_i\ a_{i+1} = -1$ if and only if the two factors are of opposite signs. In other words, k is the number of changes of signs in the squence $a_1, a_2, a_3, \ldots\ldots, a_n\ a_1$. The changes of signs from 1 to -1 are as often as those from -1 to 1.

Hence k = 2m for some integer m

ie. n = 2k = 4m

ie. $4|n$

9. Let n be a positive integer which is not divisible by 2 or 5. Prove that there is a multiple of n consisting entirely of 1's

Ans. Consider the n integers

1, 11, 111, ... , 111 ..... 1. modulo n

There are n possible remainders

0, 1, 2, ..., $(n-1)$

If the remainder 0 occurs then clearly some multiple of n will consist entirely of 1's

If not, two of the members have the same remainder modulo n.

There difference 111 .... 1000 ..... 0 is divisible by n. But n is not divisible by 2 or 5.

Therefore we can strike out the zeros in the number and the resulting number consisting of 1's only is divisible by n. *i.e.* There is a multiple of n consisting entirely of 1 only.

10. Find all pairs (x,y) of non negative integers such that x^2+3y and y^2+3x are simultaneously perfect squares.

Ans. Clearly, $x^2+3y > x^2$ and $y^2+3x > y^2$

But we must have x^2+3y and y^2+3x as perfect squares of non negative integers. Possibly, we may have

$x^2+3y \geq (x+1)^2$ and $(y^2+3x) \geq (y+1)^2$

Suppose,

$$x^2+3y \geq (x+2)^2 \text{ and } (y^2+3x) \geq (y+2)^2$$

Then $x^2+3y+y^2+3x \geq (x+2)^2+(y+2)^2$

$$\Rightarrow x^2+y^2+3(x+y) \geq x^2+y^2+4(x+y)+8$$

$$\Rightarrow x+y+8 \leq 0.$$ which is impossible.

Therefore, $x^2+3y \geq (x+2)^2$ and $y^2+3x \geq (y+2)^2$ are not possible simultaneonsly

i.e. one of the two inequalites is not true.

To be specific, let $x^2+3y < (x+2)^2$ and $y^2+3x \geq (y+2)^2$

ie. $x^2 < x^2+3y < (x+2)^2$

since x^2+3y is a perfect square

we have $x^2+3y = (x+1)^2$

ie. $x^2+3y = x^2+2x+1$

ie. $3y = 2x+1$

ie. $y = \dfrac{2x+1}{3}$

ie. $x=3k+1$ for some integer k

ie. $y=\frac{2(3k+1)+1}{3}=\frac{6k+3}{3}=2k+1$

Now, $y^2+3x = (2k+1)^2+3(3k+1)$

$= 4k^2+4k+1+9k+3$

$= 4k^2+13k+4$

For $K>5$

$$(2k+3) < 4k^2+13k+4 < (2k+4)^2$$

since $(2k+3)^2$ and $(2k+4)^2$ are consecutive perfect squares, $4k^2+13k+4$ can not be a perfect square.

Also, for $k\in\{1,2,3,4\}$ the values of $y^2+3x = 4k^2+13k+4$ are 21, 46, 79, 120 which are non perfect squares.

Also, for $k=0$, $y^2+3x = 4=2^2$ a perfect square

and for $k=5$ $y^2+3x = 169=13^2$ again a perfact square

Also, for $k=0$ and $k=5$, the values of

$x^2+3y = (3k+1)^2+3(2k+1)$

$= 9k^2+6k+1+6k+3$

$= 9k^2+12k+4$

are 4 and $225+60+4 = 289 = 17^2$ which are perfect squares.

Hence possible values of (x, y) are $(3k+1, 2k+1)$

ie. (1, 1), (16, 11) and (11, 16).

11. Let a, b, c be the sides of a triangle. Then prove that

$ab+bc+ca\le a^2+b^2+c^2\le 2(bc+ca+ab)$.

Ans : $(a-b)^2\ge 0$, $(b-c)^2\ge 0$, $(c-a)^2\ge 0$

$\Rightarrow (a-b)^2+(b-c)^2+(c-a)^2\ge 0$

$\Rightarrow a^2-2ab+b^2+b^2-2bc+c^2+c^2-2ca+a^2\ge 0$

P.T.O.

$\Rightarrow 2(a^2+b^2+c^2)-2(bc+ca+ab)\ge 0$

$\Rightarrow bc+ca+ab\ge a^2+b^2+c^2$

Also, a, b, c being sides of a triangle,

$a\ge b-c$, $b\ge c-a$ and $c\ge a-b$

so $a^2\ge (b-c)^2$, $b^2\ge (c-a)^2$ and $c^2\ge (a-b)^2$

$\therefore$ $a^2+b^2+c^2\ge (b-c)^2+(c-a)^2+(a-b)^2$

i.e. $a^2+b^2+c^2\ge 2(a^2+b^2+c^2)-2(bc+ca+ab)$

i.e. $2(bc+ca+ab)\ge a^2+b^2+c^2$

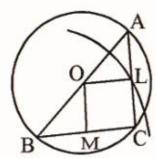
Thus $bc+ca+ab\le a^2+b^2+c^2\le 2(bc+ca+ab)$

12. Construct a triangle, given one side as a, median m of the other side, and radius R of a circumscribed circle.

Ans. Let ABC be a triangle whose one side BC= a, median m_b from B to AC and circumradius R of the circumcircle of the triangle are given.

Draw the circle of radius R with O as centre.

Draw a chord BC=a of the circle. An arc with B as centre and radius equal the m_b is drawn. Let $OM\perp BC$. Draw $CA\parallel OM$ and $OL\perp CA$, Join AB. Then our required triangle is ΔABC with m_b=BL.



13. Two boys with one bicycle between them set out from A in the direction of B, one by bicycle and the other on foot.

At a certain distance from A, the one riding the bicycle left it by the road and continued towards B on foot. The one who had started out on foot reached the bicycle and rode the rest of the distance. Both reach B at the same time. On the return trip from B to A, they did as before, but this time the cyclist rode one kilometre more than the first time and so his comrade arrived in A 21 minutes after he did. Find the rate of each of the boys on foot if they both did 20km/hour cycling and on foot, the first took 3 minutes less to cover each kilometre than the second.

Ans. Let s km be the distance between A and B, v km/hr be the rate of walking for the first boy, w km/hr be the rate of walking for the second boy. A km be the distance that the first boy cycled from A to some place in between A and B after which he walked to reach B.

Then the time taken by the first boy from A to B is

$$\frac{a}{20}+\frac{s-a}{v}\text{ hours.}$$

and the time taken by the second boy is

$$\frac{a}{w}+\frac{s-a}{20}\text{ hours.}$$

By condition,

$$\frac{a}{20}+\frac{s-a}{v}=\frac{a}{w}+\frac{s-a}{20} \quad \ldots\ldots\ldots\ldots \text{(i)}$$

For the return trip, the respective equation is

$$\frac{a+1}{20}+\frac{s-a-1}{v}=\frac{a+1}{w}+\frac{s-a-1}{20}-\frac{21}{60} \quad \ldots\ldots\ldots \text{(ii)}$$

Again, $\frac{1}{w}-\frac{1}{v}=\frac{3}{60}=\frac{1}{20} \quad \ldots\ldots\ldots\ldots \text{(iii)}$

(i)−(ii) $\Rightarrow \frac{1}{20}+\frac{1}{v}=-\frac{1}{w}+\frac{1}{20}+\frac{7}{20}$

or $\frac{1}{w}+\frac{1}{v}=\frac{9}{20} \quad \ldots\ldots\ldots\ldots \text{(iv)}$

(iii)−(iv) $\Rightarrow \frac{2}{w}=\frac{10}{20}=\frac{1}{2}$

$\therefore\ w=4$

Thus $\frac{1}{v}=\frac{1}{w}-\frac{1}{20}$

$=\frac{1}{4}-\frac{1}{20}$

$=\frac{1}{5}$

$\therefore\ v=5$

❖❖❖❖