Assam Academy of Mathematics

MATHEMATICS OLYMPIAD

September 10, 2017 (Sunday)

Category-III : For classes - IX, X & XI

Total Marks : 100 Time : 11.00 AM – 2.00 PM

1. Prove that $A_n = 5^n + 2.3^{n-1} + 1$ is a multiple of 8 for every positive integer n. 10

Ans : Rewriting A_n as –

$A_n = 5^n+(3-1)3^{n-1}+1$

$= 5^n+3^n-3^{n-1}+1$

$= (5^n+3^n)-(3^{n-1}-1)$ .... (i)

and $A_n = 5^n+(5-3)3^{n-1}+1$

$= 5.5^{n-1}+5.3^{n-1}-3^n+1$

$= 5(5^{n-1}+3^{n-1})-(3^n-1)$ ....(ii)

We note that when n is odd

5^n+3^n is divided by 5+3=8 and $3^{n-1}-1$ is divided by $3^2-1=8$

By (i), An is divisible by 8

Again, when n is even, $5^{n-1}+3^{n-1}$ is divisible by 5+3=8, and 3^n-1 is divisible by $3^2-1=8$.

P.T.O.

Hence by (ii) An is divisible by 8

By (i) and (ii) therefore, is a multiple of 8 for all positive integers n.

2. Prove that the diagonals of a quadrilateral are perpendicular if and only if the sum of the squares of one pair of opposite sides equals that of the other. 8

Ans : Let a, b, c, d be the lengths of the sides of the quadrilateral and let p,q, r, and s be the lengths of the segments of the diagorals difined by their intersections as given in the figure. If the diagonals are not perpendicular let p and r be the seqments forming an obtuse angle. Then

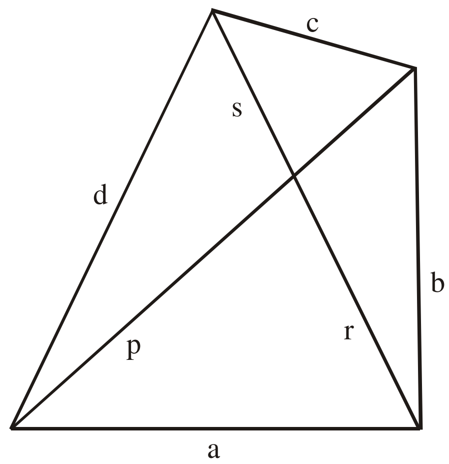
$$a^2>p^2+r^2, \quad b^2<q^2+r^2$$

$$c^2>q^2+s^2, \quad d^2<s^2+p^2$$

Hence $a^2+c^2>p^2+q^2+r^2+s^2>b^2+d^2$

which means that the sum of squares of one pair of opposite sides is not equal to that of the other.

If the liagonals are perpendicular, then by pythagoras theorem the sum of squares of one pair of opposite sides is obviously equal to that of the other.



3. Prove that for every integer n>2

$$(1.2.3 \; ....n)^2 > n^n \qquad 8$$

Ans : $(1.2.3...........n)^2$

$$= (1.n)\{2(n-1)\}\{3(n-2)\}.........\{(k+1)(n-k)\}.....(n.1)$$

Each of the bracketted products is of the form (k+1) (n-k), k=0, 1,2, ... (n-1).

The first as well as the last product is less than others because for $n-k>1$ and $k>0$

$(k+1)(n-k) = k(n-k) + (n-k) > k.1 + n - k = n$

Now the product of all these products is the expression $(1.2.3..........n)^2$ and therefore it is greater than n.n.n ....n= n^n, whenever it has more then two factors i.e. whenever n>2.

4. Prove that $11^{n+2} + 12^{2n+1}$ is divisible by 133 for any natural number n. 9

Ans : $11^{n+2}+12^{2n+1}$

$= 121.11^{n}+12.12^{2n}$

$= 133.11^{n}-12.11^{n}+12 \cdot 12^{2n}$

$\equiv 12(12^{2n}-11^{n})$ (mod 133)

$12(144^{n}-11^{n})$ (mod 133)

0 (mod 133) Since $144^{n}-11^{n}$ is divisible by 144–11=133.

Thus, $11^{n+2}+12^{2n+1}$ is divisible by 133 for any natural number n.

5. Suppose k, l and m are natural numbers

Prove that 10

Ans : For any three natural numbers k,l,m

$(2^{k}-1)(2^{l}-1)(2^{m}-1)>0$ .... (i)

Also, 2^{k+l+m}

$\geq 2^{k+2}$

$= 4-2^{k}$

$> 2^{k}+2^{k}+2^{k}$

$2^{k}+2^{l}+2^{m}$ if k l m

i.e. $2^{k+l+m}-2^{k}-2^{l}-2^{m}>0$ ....(ii)

Adding (i) and (ii),

$(-1)^{3}+(-1)^{2}(2^{k}+2^{l}+2^{m})+(-1)(2^{l+m}+2^{m+k}+2^{k+l})+2^{k+l+m}+$

$2^{k+l+m}-2^{k}-2^{l}-2^{m}>0$

i.e $2.2^{k+l+m}>2^{l+m}+2^{m+k}+2^{k+l}+1$

or $2^{k+l+m+1}>2^{l+m}+2^{m+k}+2^{k+l}+1$

6. Let *f* : N N be a function such that

(a) $f(m) < f(n)$ whenever $m < n$

(b) $f(2n) = f(n) + n$ for all n N and

$N + 2^{k+m} + 2^{l+m} \leq 2^{k+l+m+1} + 1$

(c) n is a prime number whenever $f(n)$ is a prime number. Find $f(2001)$ 10

Ans : From f (2n)= f(n)+n, it follows that

f (2)= f(1)+1

f (4)= f (2×2)=f (2)+2=f (1)+3

Let us take f (3) = f (1)+2

By induction, we have then

f (n)= f (1)+(n–1) for all n

Assume that f (1)=m>1

Then the numbers

$$|\underline{(m+1)}+2,\ |\underline{(m+1)}+3,\ \ldots\ldots\ldots\ldots,\ |\underline{(m+1)}+(m+1)$$

are all composible

Let p be the least prime exceeding $|\underline{(m+1)}+(m+1)$

Setting $n = p-m+1$, we have

$$p = n+m-1 = f(1)+n-1 = f(n)$$

By condition,

f (n) is prime implies that n is also a prime.

But $\quad n > |\underline{m+1}+2$ and hence

$$p > n > |\underline{m+1}+(m+1)$$

This contradicts the minimality of

of p. $\quad$ Hence $f(1) = m = 1$ and

Thus $f(n) = n$ for all n

Therefore $f(2001) = 2001$

7. S is a set of n positive integers. None of the elements of S is divisible by n. Prove that there exists a subset of S such that the sum of its elements is divisible by n. 10

Ans : Let us consider the sums

$$a_1,\ a_1+a_2,\ a_1+a_2+a_3,\ \ldots\ldots\ldots\ldots\ a_1+a_2+\ldots\ldots+a_n$$

where a_1, $a_{2,}$ .... a_n are n positive integers.

If any of these n sums is divisible by n then we are done. Other wise two of the sums say $a_1+a_2+\ldots\ldots+a_i$ and $a_1+a_2+\ldots\ldots+a_j$ with $j>i$ will have the same remainder upon division by n. This means

$(a_1 + a_2+\ldots\ldots\ldots +a_j) - (a_1+a_2+\ldots\ldots\ldots a_i)$ will be divisible by n.

i.e. $\quad a_{i+1}+a_{i+2}+\ldots\ldots+a_j$ will be divisible by n. Hence the solution.

8. Consider a row of n seats. A child sits on each. Each child may move at most by one seat. Find the number Fn of ways they can

∈ N rearrange. 10

Ans : Let us number the seats by 1,2,3,........,n

Let a_n be the total number of rearrangements of the children. If the first child occupies the seat 1, the number of rearrangements of the remaining children will be a_{n-1}. If the child in seat 1 goes to seat 2 then the child in seat 2 goes to seat 1, and the number of rearrangents of the remaining children will be a_{n-2}. Then clearly

$$a_n = a_{n-1}+a_{n-2}$$

Now $\quad a_1=1$

$a_2=1$

$a_3=a_2+a_1=2+1=3$

$a_4=a_3+a_2=3+2=5$

$a_5=a_4+a_3=5+3=8$

Thus $F_n=a_n$ is the n the terms in Fibonacci sequence.

9. Let P(x) be a polynomial over Z.

If p(a) = p(b) = p(c) = –1with integers a, b, c then p(x) has no integral zeros. 10

Ans : The polynomial p (x) over Z is of the form (x–a) (x–b) (x–c)q (x) –1

It z is an integral zero of p (x)

p (z) = (z–a) (z–b) (z–c) q (z)–1 =0

i.e. (z–a) (z–b) (z–c) q (z) = 1

Left hand side is product of three distinct factors while the right hand side can be product of most two distinct factors 1 and –1. Hence there is no interal zero of the polynonial.

10.

(i) Solve the inequation–

5

Ans : $x^2-|3x+2|+x \geq 0$

Let 3x+2>0 *i.e.* $x > -\frac{2}{3}$

Then the given inequation becomes

$x^2-(3x+2)+x \geq 0$

i.e. $x^2-2x-2 \quad 0$

i.e. $(x-1)^2-3 \quad 0$

i.e. $(x-1)^2 \quad 3$

i.e. x–1 or 1–x $\geq$

i.e. $x \geq 1 +$ or $1-\sqrt{3}. \geq x$

$\sqrt{3}\ -|3x+2|+x \geq 0$

i.e. 1– $\geq$ x 1+

Let 3x+2<0 i.e. $< -\frac{2}{3}$

The inequation be comes

$x^2+3x+2+x \geq 0$

i.e. $x^2+4x+2 \quad 0$

i.e. $(x+2)^2-2 \quad 0$

i.e. x+2 or –(x+2) $\geq$

i.e. $x \geq$ –2 or $-2-\sqrt{2} \geq x$

i.e. –2– $\geq$ x –2

Finally, let 3x+2=0 i.e. $x = -\frac{2}{3}$

Then the inequation becomes

$x^2+x \geq 0$

putting, , $\left(-\frac{2}{3}\right) - \frac{2}{3} = \frac{4-6}{9} < 0$

Thus $x = -\frac{2}{3}$ is not possible.

The solution of inequation is

$1-\sqrt{3} \geq x \geq 1+\sqrt{3}$ and $2-\sqrt{2} \geq x \geq \sqrt{2}-2$

(ii) Show that the number 100 .... 01 with 1961 zeros is composite. 5

Ans : 1000.......01 (1961 zeros)

$= 10^{1962}+1$

$= (10^{654})^3+1$

$= (10^{654}+1)\ \left[(10^{654})^2 - 10^{654} + 1\right]$

which is a composite number.

(iii) In the polynomial x^3+px^2+qx+r, one zero is the sum of the other two zeros. Find the relation between p, q and r. 5

Ans : Let α, β, γ be the zeros of

x^2+px^2+qx+r

so that $\alpha = \beta + \gamma$

Then,

i.e.

$\alpha\beta\gamma = -r$

i.e.

[illegible]

Also, 2 p $\alpha\beta + \beta\gamma + \gamma\alpha = q$

2

i.e.

i.e. $\left(-\frac{p}{2}\right)^2 + \frac{2r}{p} = q$

i.e. $\frac{p^2}{4} + \frac{2r}{p} = q$

or $p^3 + 8r = 4pq$