Problem Section 8

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This section contains unsolved problems, whose solutions we ask from the readers, which we will publish in the subsequent issues. All solutions should preferably be typed in LaTeX and emailed to the editor. If you would like to propose problems for this section then please send your problems (with solutions) to the above mentioned email address, preferably typed in LaTeX. Each problem or solution should be typed on separate sheets. Solutions to problems in this issue must be received by *30 December, 2022*. If a problem is not original, the proposer should inform the editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in Ganit Bikash. Solvers are asked to include references for any non-trivial results they use in their solutions.

Problem 17. *Proposed by Manjil P. Saikia (Cardiff University)*

The partitions of n with no parts divisible by ℓ are called ℓ-regular partitions, and the total number of such partitions is denoted by $b_\ell(n)$. Let us define $a_{m(\ell)}(n)$ to be the number of ℓ-regular partitions of n where the smallest part occurs at least m times. Then prove that, for all $n \geq 1$, we have

$$a_{2(2)}(n) = b_2(n) + b_2(n+1) - b_2(n+2).$$

Solutions to Old Problems

We did not receive any solutions from the readers for Problem 14. We give the solution for Problem 14 submitted by the proposer. Problems 15 and 16 are still open for solutions.

Solution 14. *Solved by the proposer. The solution below is by B. Sury (Indian Statistical Institute (Bengaluru).*

One can rewrite $f_n(t)$ as

$$f_n(t) + (t-1) = (t^n - 1) - \sum_{i=1}^{r} \frac{t^n - 1}{t^{d_i} - 1}.$$

The primitive n-th roots of unity $e^{2ik\pi/n}$ for $(k,n) = 1$ are roots of $t^n - 1$ as well as of each $\frac{t^n - 1}{t^{d_i} - 1}$ above. Thus, the integral (cyclotomic) polynomial $\Phi_n(t) := \prod_{(k,n)=1}(t - e^{2ik\pi/n})$ divides each of the polynomials $\frac{t^n-1}{t^{d_i}-1}$ and the polynomial $t^n - 1$. Hence, there are integral polynomials $g(t)$ and $g_i(t)$ for $i \leq r$ so that

$$t^n - 1 = \Phi_n(t)g(t)\ ,\ \frac{t^n - 1}{t^{d_i} - 1} = \Phi_n(t)g_i(t)\ \forall\ i \leq r.$$

So, for each integer a, the integer $\Phi_n(a)$ divides $a^n - 1 - \sum_{i=1}^{r} \frac{a^n-1}{a^{d_i}-1} = f_n(a) + a - 1$. Therefore, if $f_n(a) = 0$, the integer $a - 1$ would have a divisor of the form $\Phi_n(a) = \prod_{(k,n)=1}(a - e^{2ik\pi/n})$. We claim that

$$\prod_{(k,n)=1} |a - e^{2ik\pi/n}| > |a - 1|,$$

which would contradict the assumption that the former divides the latter (because $a \neq 1$ as $f_n(1) \neq 0$).

But, for each k with $(k,n) = 1$, we have $|a - e^{2ik\pi/n}|^2 = a^2 + 1 - 2a\cos(2k\pi/n) > a^2 + 1 - 2a$ which implies that $|\Phi_n(a)| > |a - 1|$ and we get a contradiction to the assumption that $f_n(a) = 0$. The proof is complete.

"I think there are many analogies between fishing and science. Two keys to success in fishing are technical expertise (e.g. how to bait a hook) and intuition (e.g. knowing where to fish). Both are easier to learn by apprenticeship than from a book. Similar considerations apply in science. But technical expertise and intuition only get you so far; you also need luck. On any given day the biggest fish might be caught by a novice rather than an expert. The same is true of scientific discovery."

– **William Kaelin**
Nobel Laureate physician-scientist