The Extremal Principle

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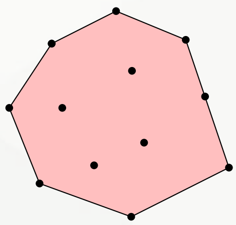
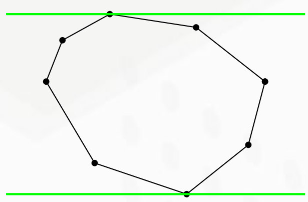
1. Introduction

The extremal principle is a problem-solving tactic, wherewith one tries to look at objects with extreme (maximal or minimal) properties. Extremal principle is best understood by solving problems. Before going into problems, understanding the notions of *convex hull* and *base lines* is necessary.

- The *convex hull* of a finite set of points is the least unique convex polygon which contains all these points. Here, the term "least" means that the polygon is not contained in any other such polygon.



- A *base line* of a convex polygon is a line passing through a vertex such that the polygon lies completely on one side of it. It is easy to verify that for any convex polygon, there exist precisely two *base lines* parallel to a given line.



It's now time to solve some problems.

2. Problems with Solutions

Problem 1: There exists no quadruple of positive integers (x, y, z, w) satisfying

$$x^2 + y^2 = 3(z^2 + w^2).$$

Solution: Suppose there exist such quadruples. Let (a, b, c, d) be a solution with the smallest value of $x^2 + y^2$. Then $a^2 + b^2 = 3(c^2 + d^2) \Rightarrow 3|a^2 + b^2 \Rightarrow 3|a, 3|b$ (Why?) $\Rightarrow a = 3a_1, b = 3b_1$ for some $a_1, b_1 \in \mathbb{Z}$. But $a^2+b^2 = 9(a_1^2+b_1^2) = 3(c^2+d^2) \Rightarrow c^2+d^2 = 3(a_1^2+b_1^2)$, i.e., (c, d, a_1, b_1) is a solution such that $c^2 + d^2 < a^2 + b^2$, a contradiction. Therefore, there exists no positive integer solutions of the given equation. □

Problem 2: Imagine an infinite chessboard that contains a positive integer in each square. If the value in each square is equal to the arithmetic mean of the values in its four neighbour squares (north, south, west and east), prove that all the positive integers are equal to each other.

Solution: Consider the smallest positive integer (say m) on the board. Since m is equal to the arithmetic mean of the values in its four neighbour squares (say a, b, c, d), therefore $a+b+c+d = 4m$. Now since m is the smallest positive integer on the board, so none of a, b, c, d can be smaller than m. Also, none of a, b, c, d can be greater than m, otherwise $a + b + c + d$ would be greater than $4m$. Therefore, each of a, b, c, d is equal to m i.e., $a = b = c = d = m$. It follows that all the positive integers on the board are equal. □

Problem 3: In a badminton singles tournament, each player played against all the others exactly once and each game had a winner. After all the games, each player listed the names of all the players she defeated as well as the names of all the players defeated by the players defeated by her. Prove that at least one player listed the names of all other players.

Solution: Assume, to the contrary, that no player listed the names of all other players. Then there is a player (say A) whose list contains maximum number of players. Since A's list does not contain the names of all other players, so A has lost against some player (say B), otherwise if A had won all the games, her list would contain the names of all other players. But this implies that B's list contain more names than that of A, a contradiction. Therefore, there is at least one player who listed the names of all other players. □

Problem 4: At a marriage party, no boy dances with every girl, but each girl dances with at least one boy. Prove that there are two girl-boy couples g_1b_1 and g_2b_2 who dance such that g_1 doesn't dance with b_2 and g_2 doesn't dance with b_1. [Putnam, 1965]

Solution: There is a boy (say b_1) who dances with most girls. Since no boy dances with every girl, so there is a girl (say g_2) with whom b_1 doesn't dance. Also, since every girl dances with at least one boy, therefore g_2 dances with at least one boy (say b_2). Now if we show that there is a girl with whom b_1 dances but b_2 doesn't, then we are done. Assume to the contrary that b_2 dances with all the girls with whom b_1 has danced. But this implies that b_2 has danced with the most number of girls (because b_2 has danced with g_2 also), a contradiction. Therefore, there is a girl (say g_1) with whom b_1 dances but b_2 doesn't. Hence, there are two couples g_1b_1 and g_2b_2 who dance such that g_1 doesn't dance with b_2 and g_2 doesn't dance with b_1. □

Problem 5: Rooks are placed on the $n \times n$ chessboard satisfying the following condition: If the square (i, j) is free, then at least n rooks are on the ith row and jth column together. Show that there are at least $n^2/2$ rooks on the board.

Solution: WLOG, we consider a row that has the least number of rooks (say m) on it. If $m \geq n/2$, then each row has at least n rooks on it and hence there are at least $mn = n^2/2$ rooks on the board. If $m < n/2$, then through the $n - m$ free squares in the row, there are $n - m$ columns which contain at least $n - m$ rooks each so that there are at least $n - m + m = n$ rooks on each column and the row together. Hence there are at least $(n - m)(n - m) = (n - m)^2$ rooks on all the columns through a free square in that row. Since each of the columns through the m non-free squares in that row has at least m rooks on it, therefore there are at least $m \times m = m^2$ rooks on all the columns through a non-free square in that row. Hence, there are at least $(n - m)^2 + m^2$ rooks on the board. Now,

$$(n-m)^2 + m^2 = \frac{n^2 + (n-2m)^2}{2} \geq \begin{cases} \dfrac{n^2}{2}, & \text{if } n \text{ is even} \\ \dfrac{n^2+1}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

Therefore, there are at least $n^2/2$ rooks on the board. □

Problem 6: Let $f(x)$ be a polynomial of degree n with real coefficients such that $f(x) \geq 0 \ \forall x \in \mathbb{R}$. If $g(x) = f(x) + f'(x) + \cdots + f^{(n)}(x)$, then show that $g(x) \geq 0 \ \forall x \in \mathbb{R}$.

Solution: Since $f(x) \geq 0$, we first try to look at the value(s) of x for which $f(x)$ is minimal. To prove that the minimal value exists, we proceed as follows. Let

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

where each $a_i \in \mathbb{R}$. Since $f(x) \geq 0$, so $a_n > 0$ because the leading term $a_n x^n$ dominates the value of $f(x)$ for large values of $|x|$. Also, the degree of $f(x)$ i.e., n must be even. Thus

$$\lim_{n \to -\infty} f(x) = \lim_{n \to +\infty} f(x) = +\infty,$$

and hence $f(x)$ has a minimum value. Since $g(x)$ has the same leading term as $f(x)$, so $g(x)$ also has a minimum value. We are to prove that $g(x) \geq 0 \ \forall x \in \mathbb{R}$. Assume, to the contrary, that $g(x) < 0$ for some value(s) of x. Let $g(x)$ attains it's minimum at $x = x_0$. Therefore, $g(x_0) < 0$. Now,

$$g'(x) = f'(x) + f''(x) + \cdots + f^{(n+1)}(x).$$

Since $f(x)$ is of degree n, so $f^{(n+1)}(x) = 0$. Therefore,

$$g'(x) = f'(x) + f''(x) + \cdots + f^{(n)}(x) = g(x) - f(x).$$

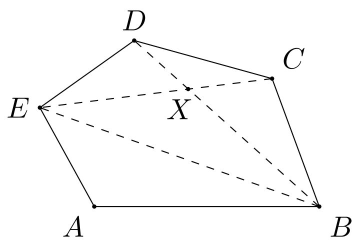
Since $g(x_0) < 0$ and $f(x_0) \geq 0$, therefore $g'(x_0) = g(x_0) - f(x_0) < 0$, which is a contradiction because by assumption $g(x)$ attains it's minimum at $x = x_0$, so $g'(x_0)$ must be equal to zero. Therefore, $g(x) \geq 0 \ \forall x \in \mathbb{R}$. □

Problem 7: Let Ω be a set of points in the plane. Each point in Ω is a midpoint of two points in Ω. Prove that Ω is an infinite set.

Solution: Suppose Ω is a finite set. Then Ω contains two points A, B with maximal distance $AB = d$. By the hypothesis, $\exists$ some $C, D \in \Omega$ such that B is the mid-point of CD. Therefore, $AC > AB$ or $AD > AB$ a contradiction to $AB = d$ being the maximal distance. □

Problem 8: Prove that in every convex pentagon, we can choose three diagonals from which a triangle can be constructed.

Solution: Let $ABCDE$ be a convex pentagon and BE be it's longest diagonal. Join BE, BD and EC.

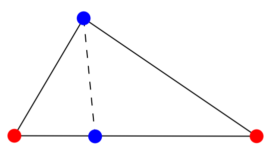


By triangle inequality in ΔBEX and ΔCDX, we have $BX + EX > BE$ and $CX + DX > CD$. Adding, we have $(BX + DX) + (EX + CX) > BE + CD \Rightarrow BD + CE > BE + CD > BE$. Hence, the sides BD, CE and BE form a triangle. □

Problem 9: Suppose there are finitely many red and blue points on a plane with the property that every line segment joining two points of the same colour contains a point of another colour. Prove that all the points lie on a single straight line.

Solution: Suppose that the points do not lie on a single straight line. Therefore, finitely many triangles can be drawn with those finitely many points as vertices. So, there exists a triangle (say T) with least area. Since there are points of two distinct colours, so at least two vertices of the triangle have the same colour.

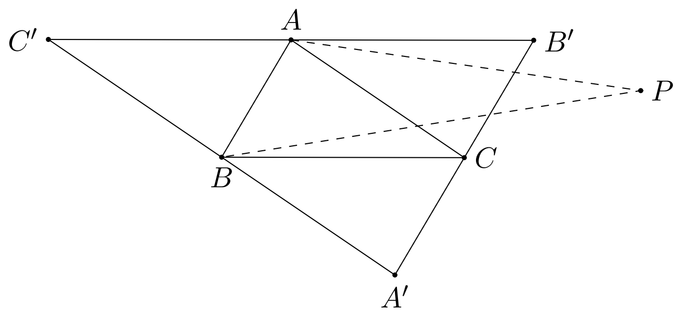
Suppose two vertices of the triangle are of same colour (say red) and the other is blue. Then there is a blue point on the line joining the two red vertices. Now the blue point is joined with the blue vertex to obtain two triangles on either side with area less than that that of T, a contradiction.



Similar argument follows if all the vertices are of same colour. Therefore, all the points lie on a single straight line. □

Problem 10: There are n points in a plane such that the area enclosed by any three of the points do not exceed 1. Prove that a triangle can be drawn with area not more than 4 that contains all n points. [Korea, 1995]

Solution: Among all the triangles formed by the n points, consider the triangle (say, ABC) with maximum area (not exceeding 1). Let $A'B'C'$ be it's anti-complementary triangle, i.e., the triangle formed by the parallels to the sides of ΔABC and passing through it's vertices. So ΔABC is the medial triangle of $\Delta A'B'C'$, with area of $\Delta A'B'C'$ not exceeding 4. Now we shall prove that $\Delta A'B'C'$ contains all n points. Assume to the contrary that $\exists$ a point (say, P) outside $\Delta A'B'C'$. WLOG, assume that the line $A'B'$ separates the point P from $\Delta A'B'C'$.



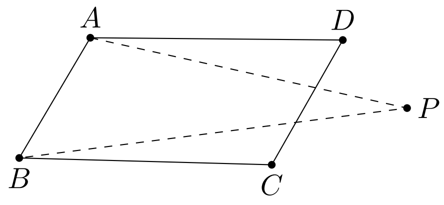
But then the area of ΔABC is greater than that of ΔABC (Why?), a contradiction. It follows that the given statement is true. $\square$

Problem 11: A strip of width w is the set of all points which lie on or between two parallel lines that are at at a distance w apart. Let S be a set of n ($n \geq 3$) points on the plane such that any three different points of S can be covered by a strip of width 1. Prove that S can be covered by a strip of width 2. [Balkan MO 2010]

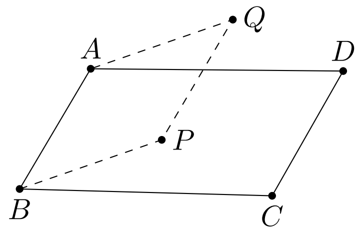
Solution: Among all the triangles formed by the n points in S, consider the triangle (say, ABC) with maximum area. Proceeding as in Problem 10, we can prove that all the points in S should lie inside the anti-complementary triangle (say $A'B'C'$) of ΔABC, otherwise we have a contradiction. By the property of S, ΔABC has altitudes of length at most 1, so $\Delta A'B'C'$ has altitudes of length at most 2 (Why?). Hence, S can be covered by a strip of length 2. $\square$

Problem 12: Find all sets S of finitely many points in the plane, no three of which are collinear and such that for any three points A, B, C in S, there is another point D in S such that A, B, C, D (in some order) are the vertices of a parallelogram. [USA TST 2005]

Solution: We claim that any such set S contains at most 4 points. Assume, to the contrary, that S has more than 4 points. Consider points A, B, C in S such that ΔABC has the maximum possible area over all choices of A, B, C. Now there is a point D such that $ABCD$ is a parallelogram. Now, any other point P in S cannot lie outside $ABCD$ because in that case, the area of ΔABP will be greater than that of ΔABP (assuming WLOG that the line CD separates P from AB).



Therefore, P lies inside $ABCD$. But then for any point Q in S such that $ABPQ$ is a parallelogram, Q lies outside $ABCD$.

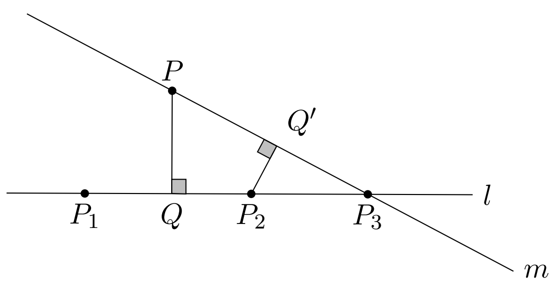


Now, the area of ΔBQC is greater than that of ΔABC (assuming WLOG that the line AD separates Q from BC), a contradiction. Therefore, any such set S contains at most 4 points. □

Problem 13: (Sylvester) Prove that every finite set of points in the Euclidean plane has a line that passes through exactly two points or a line that passes through all of them.

Solution: (L.M. Kelly) Let Ω be the finite set of points, not all collinear. Define a *connecting* line to be a line that contains at least two points of Ω, and an *ordinary* line to be a line that contains exactly two points among the *connecting* lines. Since there are finitely many points in Ω, so there are finitely many *connecting* lines. Thus, $\exists$ a point P in Ω and a *connecting* line l with minimum non-zero distance (say PQ) between them such that $PQ \perp l$. It can be proven that l is *ordinary* by contradiction as follows.

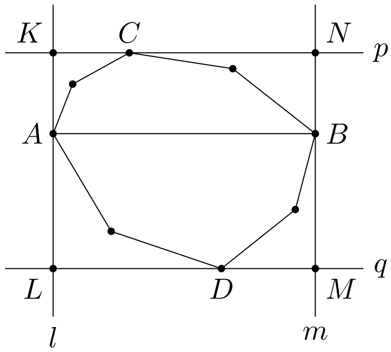
Assume that l is not *ordinary*. It follows that l contains at least three points (say P_1, P_2, P_3) of Ω. Therefore, at least two of those points (say P_2, P_3) are on the same side of Q. Let P_2 be the closest to Q (with possibly coinciding with it). Let m be the *connecting* line joining PP_3. But the perpendicular P_2Q' from P_2 to m is shorter than PQ, a contradiction to the assumption that PQ is the minimum distance between a point in Ω and a *connecting* line.



It follows that l is *ordinary*. □

Problem 14: Prove that any convex polygon of area 1 can be placed inside a rectangle of area 2.

Solution: Let AB be the greatest diagonal (or side) of the polygon. Line l and m perpendicular to AB are drawn. For any vertex X of the polygon, $AX \leq AB$ and $BX \leq AB$. Therefore, the polygon lies inside the band formed by the lines l and m. Draw the *base lines* (say p and q) of the polygon parallel to AB and let these lines pass through vertices C and D of the polygon respectively. Let $KLMN$ be the rectangle formed by the lines l, m, p and q.



Now, $[KLMN] = [KABN] + [ALMB] = 2[ABN] + 2[ABM] = 2[ABC] + 2[ABD] = 2[ABCD]$, where $[*]$ denotes the area of $*$. Now, $[ABCD] \leq 1$ since $ABCD$ is contained inside the polygon. Therefore, $[KLMN] \leq 2$, i.e., any convex polygon of area 1 can be placed inside a rectangle of area 2. □

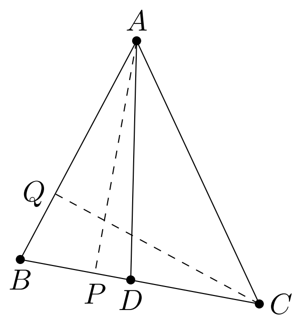
Problem 15: Prove that in the coordinate plane, the vertices of a regular pentagon cannot be all lattice points.

Solution: Suppose there exist such pentagons. Among all such pentagons, consider the one with the least area. Let $\vec{v_i} = a_i\hat{i} + b_i\hat{j}$ for $i \in \{1, 2, 3, 4, 5\}$ be the it's five side vectors that point from one vertex to the next and $a_i^2 + b_i^2 = p^2$, where p is the length of the side of the regular pentagon. Therefore, $\sum\limits_{i=1}^{5} \vec{v_i} = 0$ and hence $\sum\limits_{i=1}^{5} a_i = 0$ and $\sum\limits_{i=1}^{5} b_i = 0$. Squaring and adding, we have $\left(\sum\limits_{i=1}^{5} a_i\right)^2 + \left(\sum\limits_{i=1}^{5} b_i\right)^2 = 0 \Rightarrow \sum\limits_{i=1}^{5} a_i^2 + \sum\limits_{i=1}^{5} b_i^2 + 2\sum\limits_{1\leq i<j\leq 5}(a_ia_j + b_ib_j) = 0$. Therefore, $\sum\limits_{i=1}^{5} a_i^2 + \sum\limits_{i=1}^{5} b_i^2$ is even. Since $a_i^2 + b_i^2 = p^2$, so $\sum\limits_{i=1}^{5} a_i^2 + \sum\limits_{i=1}^{5} b_i^2 = 5p^2$. Hence p^2 is even, which implies that a_i and b_i are of same parity. If a_1 and b_1 are both odd, then $4 \nmid a_1^2 + b_1^2 = k^2$. Therefore, all a_i and b_i are odd. But then the sum of five odd numbers $\sum\limits_{i=1}^{5} a_i \neq 0$, a contradiction. If a_1 and b_1 are both even, then $4 | \, a_1^2 + b_1^2 = k^2$. Therefore, all a_i and b_i are even. But then we can scale our pentagon by $\frac{1}{2}$ to get a regular pentagon with lattice points having smaller area, a contradiction. □

Problem 16: The lengths of a triangle's bisectors do not exceed 1. Prove that the area of the triangle does not exceed $\frac{1}{\sqrt{3}}$.

Solution: Let $\angle A$ be the smallest angle of ΔABC. Therefore, $\angle A \leq 60°$, otherwise $\angle A$ is no longer the smallest angle. Let AD be the angle bisector of $\angle A$ such that D lies on BC. Draw $AP \perp BC$

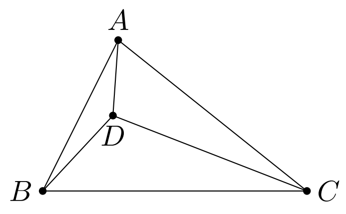
and $CQ \perp AB$ such that P and Q lie on BC and AB respectively. WLOG, suppose P lies between B and D (with possibly coinciding with B or D).



Now, $\cos\angle BAP = \frac{AP}{AB}$. Since $\angle BAD \geq \angle BAP$, therefore $\cos\angle BAD = \cos\frac{A}{2} \leq \cos\angle BAP \leq \frac{AD}{AR} \leq \frac{AD}{AB}$ (assume R to be on extended AB such that $RD \perp AD$, therefore $AR \geq AB$), i.e., $AB \leq \dfrac{AD}{\cos\frac{A}{2}} \leq \dfrac{AD}{\cos 30^\circ} \leq \dfrac{1}{\frac{\sqrt{3}}{2}}$, i.e., $AB \leq \dfrac{2}{\sqrt{3}}$. Since perpendicular is the shortest distance from a point to a line, so $CQ \leq$ the angle bisector of $C \leq 1$. Therefore, $[ABC] = \dfrac{1}{2}AB \times CQ \leq \dfrac{1}{2}AB \times 1 \leq \dfrac{1}{2} \times \dfrac{2}{\sqrt{3}}$,i.e., $[ABC] \leq \dfrac{1}{\sqrt{3}}$. Thus, if the lengths of a triangle's bisectors do not exceed 1, then the area of the triangle does not exceed $\frac{1}{\sqrt{3}}$. □

Problem 17: Given four points in plane not on one line. Prove that at least one of the triangles with vertices in these points is not an acute one.

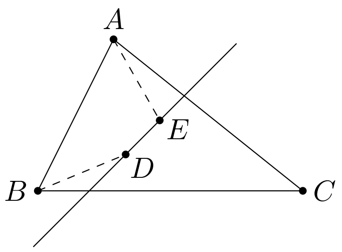
Solution: Consider the convex hull of the four points. If it is a quadrilateral (say $ABCD$) and $\angle ABC$ is the largest angle, then $\angle ABC \geq 90^\circ$ and hence ΔABC is not acute. Let ΔABC be the convex hull and D be a point in it's interior. WLOG, let $\angle ADB$ be the largest among $\angle ADB$, $\angle BDC$ and $\angle CDA$. Therefore, $\angle ADB \geq 120^\circ$ and hence ΔADB is not acute.



Hence, the given four points form at least one triangle that is not acute. □

Problem 18: (Happy Ending Problem) Five points in the plane are given, no three collinear. Prove that some four of them form a convex quadrilateral.

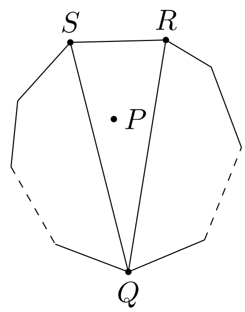
Solution: Consider the convex hull of the points. If it is a pentagon or a quadrilateral (with one point in it's interior), then it is clearly true. If the convex hull is a triangle (with two points in it's interior), then it can be proven as follows. Let ΔABC be the convex hull and D and E be two points in it's interior. DE is joined and extended on both sides. Hence two vertices of the triangle (WLOG, say A and B) lies on one side of extended DE. Then points A, B, D and E form a convex quadrilateral.



Hence, four among the five given points form a convex quadrilateral. □

Problem 19: There are $n \geq 3$ coplanar points, no three of which are collinear and every four of them are the vertices of a convex quadrilateral. Prove that the n points are the vertices of a convex n-sided polygon.

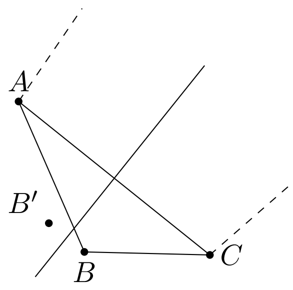
Solution: Consider the convex hull W of the n points. So all the n points are either inside or on W. If we prove that no point lies inside W, then W will be the required convex n-sided polygon and we are done. Assume, to the contrary, that some point P among those n points lies inside W. If Q is any vertex of W, then there exist vertices R and S of W such that P lies completely inside ΔQRS. But then $PQRS$ is not a convex quadrilateral, a contradiction.



Hence, W is the n-sided polygon on which all the n points lie as vertices. □

Problem 20: A set S of points in the plane will be called *completely symmetric* if it has at least three elements and satisfies the following condition: For every two distinct points A, B from S the perpendicular bisector of the segment AB is an axis of symmetry for S. Prove that if a completely symmetric set is finite then it consists of the vertices of a regular polygon. [IMO 1999 P1]

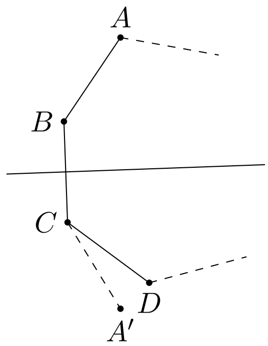
Solution: Consider a convex hull W of the set of points in S with n sides ($n \geq 3$).
Claim 1: All sides of W have equal length.
Proof: Assume to the contrary that W has two sides AB and BC of unequal length. WLOG, suppose $AB > BC$. But then the perpendicular bisector of AC reflects B to B' such that B' lies outside W, a contradiction.



Therefore, all sides of W have equal length.

Claim 2: All angles of W are also equal and hence W is a regular polygon.

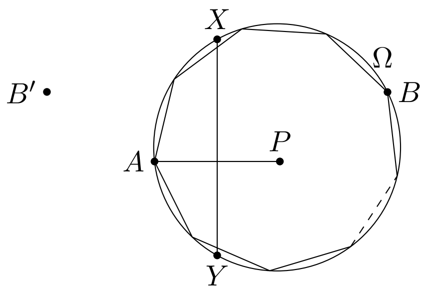
Proof: For $n = 3$, it is obvious because if all sides of a triangle are equal, then the angles are also equal. Take $n > 3$ and assume to the contrary that A, B, C, D are four points of W such that $\angle ABC \neq \angle BCD$. WLOG, suppose $\angle ABC > \angle BCD$. But then the perpendicular bisector of BC reflects A to A' such that A' lies outside W, a contradiction.



Therefore, W is a regular polygon.

Claim 3: No point of S lies inside W i.e., all points of S lie on W.

Proof: Assume, to the contrary that $\exists$ a point P of S lying inside W. Let Ω be the circumcircle of W and join P with any vertex A of the polygon. Let the perpendicular bisectors of AB meet Ω at X and Y. Let B be any vertex of the polygon lying on the major arc of XY. But then XY reflects B to B' such that B' lies outside Ω and hence outside W, a contradiction.



Therefore, all points of S lie on W. Hence, if a completely symmetric set is finite then it consists of the vertices of a regular polygon. □

3. Practice Problems with Hints

Problem 21: There are finitely many points on a circle, and each point is given a positive integer that is equal to the average of the numbers of its two nearest neighbours. Prove that all the positive integers are equal.
Hint: This is similar to Problem 2. Consider the smallest positive integer m. Then $m = a + b$, where a and b are it's nearest neighbours, etc.

Problem 22: Given n points in the plane, no three collinear, prove that there is a polygon with all n points as its vertices.
Hint: Consider the path with the shortest length (say $P_1P_2 \cdots P_nP_1$). If this path has no self-intersections, then it is a polygon. Assume, to the contrary, that the segments P_iP_{i+1} and P_jP_{j+1} intersect at X. Apply triangle inequality and prove that replacing these segments with P_iP_j and $P_{i+1}P_{j+1}$ gives a path with a shorter length, etc.

Problem 23: In the plane, there are given finitely many pairwise non-parallel lines such that through the intersection point of any two of them one more of the given line passes. Prove that all these lines pass through one point.
Hint: Assume, to the contrary, that not all lines pass through one point. Let the least non-zero distance from an intersecting point P to a line l be PQ. At least three of those lines (say l_1, l_2, l_3) pass through P and they intersect l at P_1, P_2, P_3 (say). Then proceed similar to Problem 13.

Problem 24: Given $n \geq 3$ points on the plane not all of them on one line. Prove that there is a circle passing through three of the given points such that none of the remaining points lies inside the circle.
Hint: Consider points A and B such that AB is minimal. Are there points inside the circle with AB as diameter? Any point C with maximal $\angle ACB$ lies on the circle, etc.

Problem 25: Let A be a set of $2n$ points in the plane, no three collinear. Suppose that n of them are coloured red, and the remaining n blue. Prove or disprove: there are n straight line segments, no two with a point in common, such that the endpoints of each segment are points of A having different colours. (Putnam, 1979)
Hint: This can be proved as follows. Consider the configuration with the minimal total path length. Consider two line segments P_iP_{i+1} and P_jP_{j+1} intersecting (i.e., having a point in common) at X and proceed as in Problem 22.

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