Distances 4 Curves in the Curve Graph of Closed Surfaces

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My recent research article, *'Distance* 4 *Curves on Closed Surfaces of Arbitrary Genus'*, published in *'Topology and its Applications'* [7], is a joint work with my doctoral advisor, Dr. Sreekrishna Palaparthi. Let S_g be an orientable, closed surface of genus $g \geq 2$ and let $\mathcal{C}(S_g)$ be its associated curve graph. In [7], we give a method to build pairs of curves at a distance 4 in $\mathcal{C}(S_g)$ from pairs curves at a distance 3 in $\mathcal{C}(S_g)$. This collection of curves are the first examples of curves at a distance 4 in $\mathcal{C}(S_{g\geq 4})$. As an application, we calculate a quadratic upper bound on the geometric intersection number of curves at a distance 4 in $\mathcal{C}(S_{g\geq 4})$.

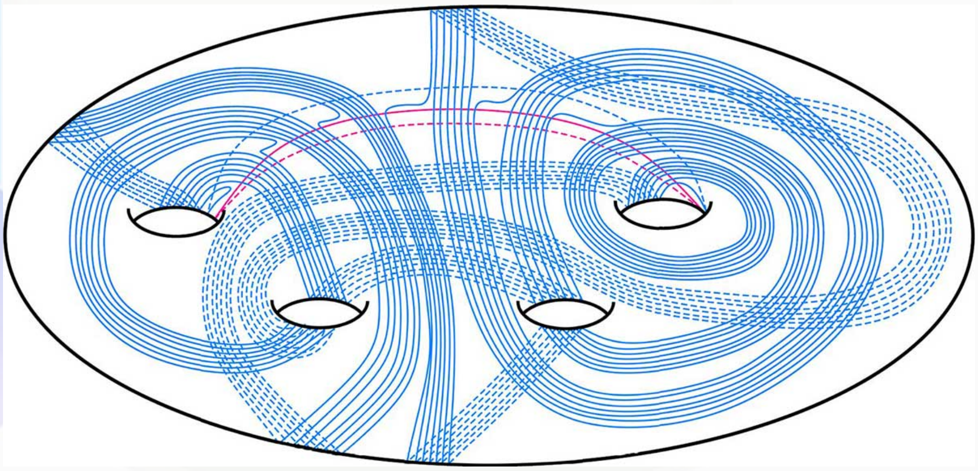

Figure 1. Curves at a distance 4 on S_4 with intersection number 49.

I have tried to include pictures to aid the understanding of certain mathematical terms. However, to aid readers irrespective of their expertise, the following is a curated list of books that can be treated as 'mathematical dictionaries' for the purpose of this article. For readers seeking a comprehensive guide in getting started in the field of topology, I have included a few book recommendations at the end of this article.

(i) For point-set topology, *Topology: A First Course* by James R. Munkres is a classic.

(ii) For a working knowledge on manifolds, one can refer to the initial part of chapter 1 of *Introduction to Smooth Manifolds* by John M. Lee.

(iii) Although not necessary for reading this article, one can consult *Hyperbolic Geometry* by James W. Anderson for an introduction to hyperbolic geometry.

(iv) For complete details on most of the remaining definitions and theorems, *A Primer on Mapping Class Groups* by Benson Farb and Dan Margalit is a cult favourite.

1. Introduction

*1.1. **Surfaces***

By a surface we will consider an orientable, connected, closed 2 dimensional $\mathbb{R}-$ manifold. In particular, a 2 dimensional $\mathbb{R}-$ manifold, M, is a second-countable, Hausdorff topological space such that around every point in M there is an open set that is homeomorphic to $\mathbb{R}^2$. Surfaces can be thought of as nice enough objects stitched from fabric into shapes we encounter on most of our days. Even if we let our imagination run wild, the following theorem assures us that the shape of a general topological surface is quite domesticated. The classification of surfaces theorem by Poincaré states that any surface is homeomorphic to the connected sum of a sphere with $g \geq 0$ tori (see figure 2). The number g is called the genus of the surface. The connected sum of two surfaces, M and N, is the surface obtained by gluing ($M \setminus$ unit disc) and ($N \setminus$ unit disc) about their boundary circles by a homeomorphism. Throughout this article we will use S_g to denote an arbitrary surface of genus $g \geq 2$. We define a curve, α, on S_g to be an embedding of the unit circle into S such that α is not null homotopic on S_g. On a lighter note, our curves can be thought of as closed elastic bands on our stitched fabrics such that the bands can't be squeezed to a point on our fabric. The moving of a curve on S_g without breaking or contracting it is called an isotopy. More precisely, an isotopy between two curves, a and b, on S_g is a continuous function, $H : S^1 \times [0,1] \longrightarrow S_g$ such that $H(S^1, 0) = a$, $H(S^1, 1) = b$ and $H(S^1, t)$ is an embedding for every $t \in [0, 1]$. By a slight abuse of notation we will write α to denote the curve, α, or the isotopy class of α whenever the context is clear. Pictorially speaking, we will consider any curve and any variation of it obtained by wriggling it as the same curve. Whenever considering a collection of isotopy classes of curves, it is a standard practice in this subject to consider minimally intersecting curves as representatives.

*1.2. **Mapping class groups***

For any S_g, its group of orientation preserving homeomorphisms (upto homotopy) is called the *mapping class group of* S_g and is denoted by $Mod(S_g)$. A class of interesting infinite order mapping class in $Mod(S_g)$ is the class of Dehn twists. Max Dehn introduced these maps and called them 'schraubungen', which translates to 'screw map'. Dehn twists in $Mod(S_g)$ can be thought of as analogous to elementary matrices in linear groups.

Figure 2. Surfaces with genus 0 (sphere), 1 (torus) and 3.

Consider the annulus, $A = S^1 \times [0,1]$, and define $T : A \longrightarrow A$ as $(\theta, r) \mapsto (\theta + 2\pi r, r)$. Let α be a curve on S_g. Let N be an annular neighbourhood of α and $\phi : A \longrightarrow N$ be an orientation preserving homeomorphism. Then, the *Dehn twist about* α, $T_\alpha : S_g \longrightarrow S_g$, is defined as follows (also, see figure 4)

$$T_\alpha(x) = \begin{cases} \phi \circ T \circ \phi^{-1}(x) & x \in N \\ x & x \notin N \end{cases}$$

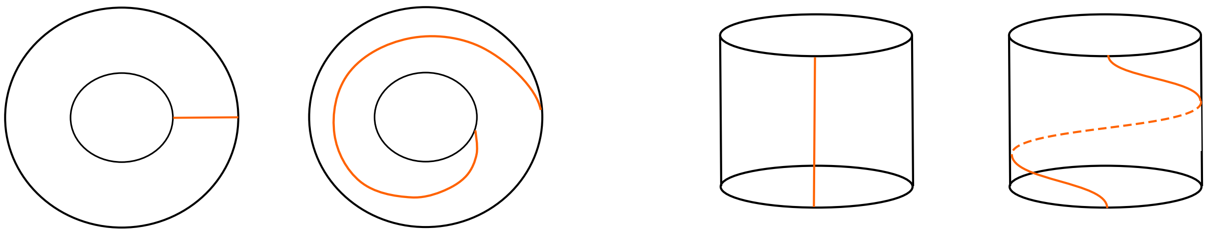

Figure 3. Annular and cylindrical view of the action of T.

The action of T_α on S_g can be interpreted as 'T acting on N' and keeping $S_g \setminus N$ fixed. The mapping class, T_α, is well-defined upto isotopy for the isotopy class of α.

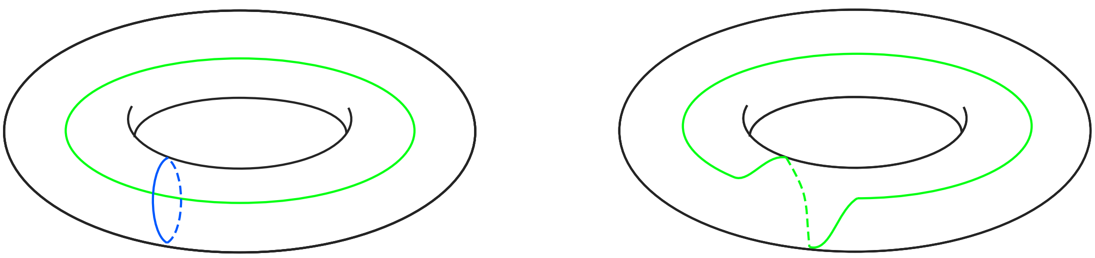

Figure 4. Dehn twist of the green curve about the blue.

1.3. ***Teichmüller spaces***

For our purpose we upgrade the topological structure of S_g to a geometrical one by equipping S_g with a Reimannian metric. From the uniformization theorem it follows that S_g is compatible with constant -1 curvature metrics. We refer to such a metric on S_g as a hyperbolic metric on S_g. One way to imagine hyperbolic structures on S_g is to consider the quotient spaces of the hyperbolic plane, $\mathbb{H}^2$, by discrete subgroups of the isometry group of $\mathbb{H}^2$ which are isomorphic to the first fundamental group of S_g. A cartoon picture of such a space might be to imagine a 'nice' enough group rolling up $\mathbb{H}^2$ such that the points which are equivalent under the action of this group are folded on top of one another. The space of marked hyperbolic metrics (upto isotopy) on S_g is called the *Teichmüller space of* S_g and is denoted by $Teich(S_g)$. It can be observed that $Mod(S_g)$ acts on $Teich(S_g)$ by the action of pullback of the hyperbolic metrics on S_g.

1.4. ***Moduli spaces***

An ubiquitous quest in any branch of mathematics is for the classification of objects upto automorphisms. For a class of geometric objects, A, its space of geometric solutions to the classification problem is called the *moduli space of* A. In this article we will restrict our interest to the moduli space of Riemann surfaces homeomorphic to any given S_g. We denote this moduli space by $\mathcal{M}(S_g)$. We can define the moduli space of hyperbolic surfaces homeomorphic to S as the quotient space

$$\mathcal{M}(S) = Teich(S)/Mod(S).$$

Studying the geometric structure on $\mathcal{M}(S_g)$ also reveals information about the geometric structures on S_g. Besides being a fundamental object in various fields ranging from low-dimensional topology, algebraic geometry to mathematical physics, $\mathcal{M}(S)$ plays a vital role in the classification of surface bundles.

1.5. ***Curve graphs***

With the motive to study the Teichmüller space, in [6] Harvey associated a simplical complex to $Teich(S_g)$, called the complex of curves. The 1-complex, $\mathcal{C}(S_g)$, of the complex of curves is called the curve graph and is defined as follows: The 0-skeleton, $\mathcal{C}^0(S_g)$, of $\mathcal{C}(S_g)$ is in one-to-one correspondence with isotopy classes of curves on S_g. Two vertices span an edge in $\mathcal{C}(S_g)$ if and only if these vertices have mutually disjoint representatives.

Define a metric, d, in $\mathcal{C}(S_g)$ such that the distance between any two vertices is the minimum number of edges in any edge path between them in $\mathcal{C}(S_g)$. The curve graph of S_g forms a connected graph ([9]). By the distance between two curves on S_g, we will mean the distance between the corresponding vertices in $\mathcal{C}(S_g)$ with respect to d. We denote the minimal geometric intersection number between any two curves on S_g which are at a distance n by $i_{min}(g, n)$.

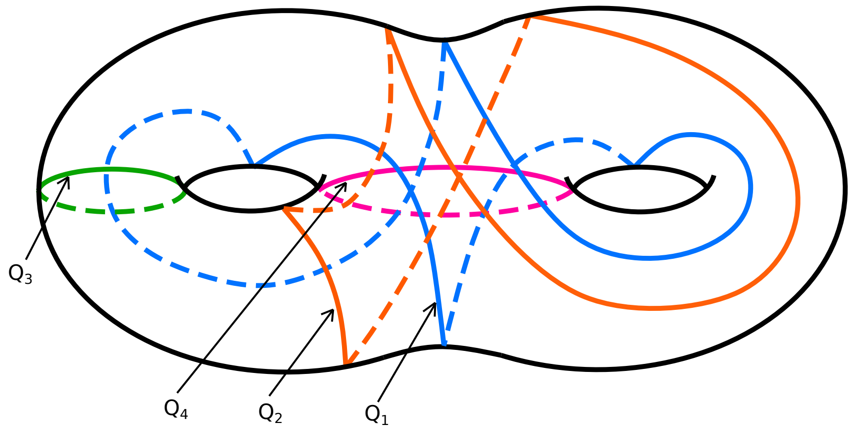

Figure 5. Q_1, Q_2, Q_3, Q_4 represents vertices in the same order of a length 3 geodesic in $\mathcal{C}(S_2)$.

The large scale geometry of the curve complex has been employed to understand the hyperbolic structure of 3-manifolds, the mapping class group and the Teichmuller space of surfaces.

2. Distances in curve graphs

Masur and Minsky proved in [9] that $\mathcal{C}(S_g)$ is $\delta-$ hyperbolic and is of infinite diameter. Later it has been proved that δ can be chosen independent of the surface S_g. Although the large scale geometry of the complex of curves is known, much about its small scale geometry remains obscure.

It can be easily observed that every vertex in $\mathcal{C}(S_g)$ has infinitely many adjacent vertices. This local infinitude of $\mathcal{C}(S_g)$ poses as a major pathology in the study of the curve graph as it hinders the calculation of distances in $\mathcal{C}(S_g)$. The authors in [8] circumvented the local infinitude of $\mathcal{C}(S_g)$ by considering a finite set of geodesics, called tight geodesics, between any two vertices. Later the authors of [3] defined another class of geodesics in $\mathcal{C}(S_g)$ called initially efficient geodesics. They proved that between any two vertices, ν and ω, at a distance n there exist finitely many geodesics $\nu_0 = \nu, \nu_1, \ldots, \nu_n = \omega$ where ν_1 can have at most n^{6g-6} possibilities.

Another hindrance in the study of the curve graph is the obscurity of possible pair of curves on S_g which represented vertices in $\mathcal{C}(S_g)$ that are a distance n apart. Using the bounded geodesic image theorem from [8], Shackleton have constructed vertices in $\mathcal{C}(S_g)$ which are a given distance apart ([11]). However the intersection number of these pairs of curves can be arbitrarily large and hence, might be somewhat not nice to study. The authors of [2] give infinite geodesic rays in $\mathcal{C}(S_g)$ such that the intersection number between the vertices of these geodesic rays is bounded above by a polynomial of the complexity of the surface and thus, is kept asymptotically low.

In [1], the authors provide an upper bound and a lower bound on the number of $Mod(S_g)-$ orbits of pairs of distance 3 curves on S_g. They further provide an algorithm to build a pair of distance 3 minimally intersecting curves on S_{g+2} from a pair of distance 3 minimally intersecting curves on $S_{g\geq 3}$. Using this the authors show that the theoretical lower bound for $i_{min}(g, 3)$ deduced from

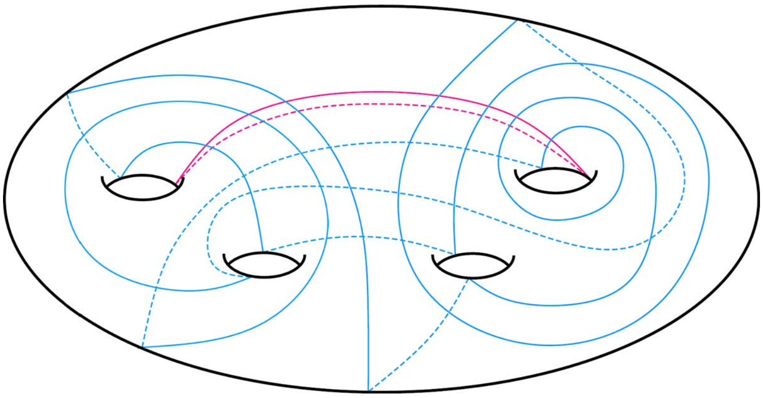

Figure 6. Curves at a distance 3 in $\mathcal{C}(S_4)$ used to obtain the pair of distance 4 curves in figure 1.

Euler characteristic considerations is obtained and hence, $i_{min}(g, 3) = 2g - 1$.

For $g \geq 3$, not only pairs of distance 4 curves remain to be elusive but also $i_{min}(g, 4)$ remains unknown. In [5], the authors prove that $i_{min}(2, 4) = 12$ by providing all pairs of distance 4 curves on S_2 in a disc with handle presentation. They prove that these curves are at a distance 4 using a software, called MICC, which runs a distance 4 test algorithm based on the efficient geodesics in [3]. Using the example of a pair of distance 4 curves on S_3 provided in [10], it has been deduced that $i_{min}(3, 4) \leq 21$.

3. Distance 4 curves in the curve graph

While trying to figure how $\mathcal{C}(S_g)$ looked like locally, the obscurity surrounding the small scale geometry of $\mathcal{C}(S_g)$ motivated Dr. Sreekrishna and me to figure out the nuts and bolts of $\mathcal{C}(S_g)$. In [7], we establish our first step of this venture by providing the first set of infinitely many examples of pairs of distance 4 curves on $S_{g \geq 4}$. These curves are a result of the following theorem 1.

Theorem 1. If α and γ are a pair of curves on S_g with $d(\alpha, \gamma) = 3$ then for $p \geq 1$, $d(\alpha, T_\gamma^p(\alpha)) = 4$.

The proof of theorem 1 can be broken down into three main steps :

Step 1 : We establish a path of length 4 between α and $T_\gamma^p(\alpha)$ by using a geodesic between α and γ. This gives that $d(\alpha, T_\gamma^p(\alpha)) \leq 4$. This brings down our choices for $d(\alpha, T_\gamma^p(\alpha))$ to either 2, 3 or 4.

Step 2 : As a second step we eliminate the possibility of 2 for $d(\alpha, T_\gamma^p(\alpha))$. We arrive at this by proving theorem 2 which is equivalent to saying $d(\alpha, T_\gamma^p(\alpha)) \geq 3$. Two curves, a and b, on S_g fill S_g if and only if $d(a, b) \geq 3$ follows from the fact that if $d(a, b) = 2$ then there is a non-trivial curve in $S_g \setminus (a \cup b)$.

Theorem 2. If α and γ are a pair of curves which fill S_g, then α and $T_\gamma^p(\alpha)$ also fill S_g for $p \geq 1$.

The core idea behind the proof of theorem 2 is that no two distinct components of $S_g \setminus (\alpha, \gamma)$ get glued to form any component of $S_g \setminus (\alpha, T_\gamma^p(\alpha))$. Each component of $S_g \setminus (\alpha, \gamma)$ disintegrates to give distinct components of $S_g \setminus (\alpha, T_\gamma^p(\alpha))$. This prevents any non-disc components from occurring in $S_g \setminus (\alpha, T_\gamma^p(\alpha))$.

Step 3 : As a final step we use theorem 3 which gives a criterion for detecting vertices in $\mathcal{C}(S_g)$ at distance at-least 4 to establish that $d(\alpha, T_\gamma^p(\alpha)) \geq 4$. The core idea executed at this step is that any minimally intersecting curve at a distance 1 from $T_\gamma^p(\alpha)$ 'takes the form of γ' while traversing on S_g. This makes such curves 'mimic' the property that γ fills S_g along with α.

Theorem 3 (Theorem 1.3, [5])**.** Let v, w be vertices in $\mathcal{C}(S_g)$ with $d(v, w) \geq 3$. Let $\Gamma \subset \mathcal{C}^0(S_g)$ be the collection of all vertices such that the following hold :

(i) for $\overline{\gamma} \in \Gamma$, we have $d(v, \overline{\gamma}) = 1$; and

(ii) for $\overline{\gamma} \in \Gamma$; there exists representatives α, β, γ of v, w, $\overline{\gamma}$ respectively, such that for each segment $b \subset \beta \setminus \alpha$ we have $|\gamma \cap b| \leq 1$.

Then $d(v, w) \geq 4$ if and only if $d(\overline{\gamma}, w) \geq 3$ for all $\overline{\gamma} \in \Gamma$.

Thus we conclude the outline of the proof of theorem 1.

Let the geometric intersection number between two curves, a and b, on S_g be denoted by $i(a, b)$. Using theorem 1, we calculate an upper bound for $i_{min}(g, 4)$. The proof of corollary 1 follows from the following two results : $i_{min}(g, 3) = (2g - 1)$ and $i(\alpha, T_\gamma^p(\alpha)) = |p| i(\alpha, \gamma)^2$.

Corollary 1. *For a surface of genus $g \geq 3$, $i_{min}(g, 4) \leq (2g - 1)^2$.*

4. Research directions

The area of mapping class group of surfaces is a vibrant and fertile research area currently. With a rich history and promising future, research in $Mod(S_g)$ is not only popular among low-dimensional geometers and topologists but many algebraists and number theorists take an active interest as well. The book, *Problems on Mapping Class Groups and Related Topics*, by Benson Farb is an excellent starting point to get introduced to various research topics in the field of $Mod(S_g)$. The book also contains many open questions. An online version of the pdf is available for free (see [4]).

References

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5. Book recommendations

The following is a list of books that I referred to through my journey into learning and re-learning topology and its related areas. I have tried to arrange the list in what I feel is an increasing level of difficulty. Lastly, I would like to convey to the reader that learning is a personal ongoing process and the list below is highly incomplete.

(i) Algebraic Topology

- Topology of Metric Spaces by S. Kumaresan
- Basic Topology by M.A. Armstrong
- Elements of Algebraic Topology by James Munkres
- Homology Theory: An Introduction to Algebraic Topology by James W. Vick
- Algebraic Topology by Allen Hatcher

(ii) Differential Topology

- Differential Geometry of Curves and Surfaces by Manfredo Perdigão do Carmo
- Geometry from a Differentiable Viewpoint by John McCleary
- First Steps in Differential Geometry: Riemannian, Contact, Symplectic by Andrew McInerney
- Calculus on Manifolds by Michael Spivak
- A Comprehensive Introduction to Differential Geometry (Vol. 1) by Michael Spivak

For comments and queries, one can write to me at *kuwari.mahanta@gmail.com*.