Logarithms and Hyperbolic Area

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In this article, I will talk about a very simple but interesting object: Logarithms. I will summarize some interesting links between logarithm and hyperbolic area and a few properties of the logarithm. First of all, the term 'logarithm' was first coined by John Napier (1550-1617), early in the seventeenth century. However it was his associate Henry Briggs (1561-1631), who constructed the familiar table of 'common log' to the base 10 and started the log table by setting $\log 1 = 0$ and $\log 10 = 1$. A log table was in its day what the computer is in the modem era. In fact, the impetus that the discovery logarithms gave to astronomical calculations cannot be over-emphasized.

Now, how and why log functions come?

The log function is an inverse function of the exponential function of the form:

$$a^x = y, \quad a > 1.$$

Here, for a fixed value of a and different values of x, we will get different values of y. But for a fixed value of a and some given value of y, how will we get the value of x? To answer this question, the log function comes into play and we can get the value of x from the log function

$$\log_a y = x.$$

And this log function was defined by Euler. In usual notation, we write

$$x = \log_a y \iff a^x = y, a > 1.$$

He also stated a rule which is known as the "golden rule for logarithm" which tells us that if we can calculate $\log_a y$, then it is an easy task to find $\log_b y$, where b is an arbitrary base. This rule is very simple as well as powerful.

Now, let us see how can we calculate the value $\log_b y$ when $\log_a y$ is given. Let the value of $\log_a y$ be x

$$x = \log_a y. \tag{1}$$

That is,

$$a^x = y$$

Take log to the base b on both sides.

$$\log_b a^x = \log_b y \implies x \log_b a = \log_b y$$

Therefore, from (1), we get

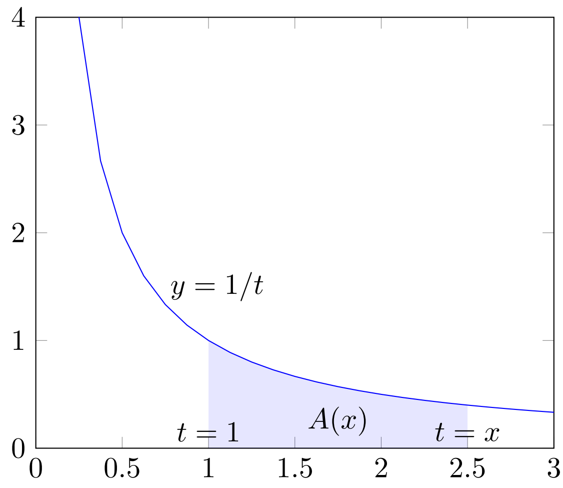
$$\log_a y \cdot \log_b a = \log_b y \implies \log_b y = \log_a y \cdot \log_b a.$$

Thus, we have got the value of $\log_b y$ by Euler's "golden rule for logarithm".

Euler quickly observed that the ratio of the logarithm of two numbers is always the same. That means, it is independent of the base.

$$\frac{\log_a y}{\log_a x} = \frac{\log_b y/\log_b a}{\log_b x/\log_b a} = \frac{\log_b y}{\log_b x}$$

Now, let us discuss a very interesting link between logarithms and hyperbolic area, which was suggested by Gregory of St. Vincent (1584-1667) and Alfonso de Sarasa (1618-1667). Let us consider a hyperbola $y = \frac{1}{t}$, and let $A(x)$ be the area of the region bounded by the hyperbola, the t-axis, $t = 1$ and $t = x$.



Then,

$$A(x) = \int_1^x \frac{1}{t} dt.$$

Now let $x = ab$, then

$$A(ab) = \int_1^{ab} \frac{1}{t} dt = \int_1^a \frac{1}{t} dt + \int_a^{ab} \frac{1}{t} dt.$$

First consider the second integral $\int_a^{ab} \frac{1}{t} dt$. Putting $t = au \implies dt = adu$ and

$$t = a \implies u = 1,$$

$$t = ab \implies u = b.$$

Therefore,

$$\int_a^{ab} \frac{1}{t} dt = \int_1^b \frac{1}{au}(adu) = \int_1^b \frac{1}{u} du,$$

$$\text{and } A(ab) = \int_1^{ab} \frac{1}{t} dt = \int_1^a \frac{1}{t} dt + \int_1^b \frac{1}{t} dt.$$

That is

$$A(ab) = A(a) + A(b).$$

In a similar way, if we find $A(a^r)$, then $A(a^r) = \int_1^{a^r} \frac{1}{t} dt$. Here, putting $t = u^r \implies dt = ru^{r-1} du$ and

$$t = 1 \implies u = 1,$$

$$t = a^r \implies u = a.$$

Therefore

$$A(a^r) = \int_1^{a^r} \frac{1}{t} dt = \int_1^a \frac{1}{u^r}(ru^{r-1} du) = r \int_1^a \frac{1}{u} du.$$

That is,

$$A(a^r) = rA(a).$$

Thus, these properties of the hyperbolic area, namely $A(ab) = A(a) + A(b)$ and $A(a^r) = rA(a)$ are nothing but the mirror of the corresponding properties of logarithms.

For more on this as well as on other amazing discoveries of Euler, we recommend the masterful treatment by Dunham [Dun99].

References

[Dun99] William Dunham. *Euler: the master of us all*, volume 22 of *The Dolciani Mathematical Expositions*. Mathematical Association of America, Washington, DC, 1999.