The Irrationality of the Golden Ratio

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The Golden Ratio, most commonly denoted by φ has fascinated people for centuries, giving rise to some mathematics (and lots of conspiracy theories). We can define this number by a simple geometrical construct. The Golden Ratio (henceforth, just referred to as φ) is the ratio of the side lengths of a rectangle with the following property: if we cut a square off the rectangle then we are left with a smaller (rotated) rectangle of exactly the same shape as the original rectangle. An easy (and, mathematical) way of seeing that φ indeed exists is to calculate the ratio.

Let us assume that the side length of the square is 1, then clearly the side lengths of the larger rectangle are 1 and φ and that of the smaller one are $\varphi - 1$ and 1. If the rectangles are of the same shape, then we have the following equation

$$\frac{\varphi}{1} = \frac{1}{\varphi - 1} \Rightarrow \varphi(\varphi - 1) = 1 \Rightarrow \varphi^2 - \varphi - 1 = 0.$$

Solving this quadratic equation we get $\varphi = \dfrac{1+\sqrt{5}}{2}$.

Now that we have calculated φ, we want to show that it is irrational. Let us now take rectangle of side lengths φ and 1. We do the following simple process: we cut off a square from it, leaving a smaller rectangle which by the definition of φ has the same shape as the original rectangle. We repeat this process over and over again, obtaining a sequence of smaller and smaller rectangles each with the same shape as the one in the preceding stage, and hence with side lengths whose ratio is φ. Clearly, this is a never-ending process.

Next, let us do a similar process with a rectangle with side lengths in the ratio $\dfrac{p}{q}$, where p, q are natural numbers. This rectangle can be divided into a square grid with $p \times q$ unit squares. Let us now remove a square from this grid (like the process we described before). For the sake of simplicity, let us assume that $q < p$, and we remove a $q \times q$ square and end up with a $q \times (p - q)$ rectangle. We can do this process one step further, but like before we cannot keep on doing this indefinitely. We can do this process at most a $p \times q$ times, because there are only $p \times q$ unit squares to begin with.

We have shown the following result.

Proposition 1. *(i) If the ratio of the sides of the rectangle we started with is φ, then we can keep removing squares forever.*

(ii) If the ration of the sides of the rectangle is $\frac{p}{q}$ for some natural numbers p and q, then we cannot keep removing squares forever.

From this proposition, it follows immediately that $\varphi \neq \frac{p}{q}$, for natural numbers p and q. And hence, φ is irrational.

Remark 2. This proof (along with many other interesting tidbits about mathematics) can be found in the short book by Gowers [Gow02].

References

[Gow02] Timothy Gowers. *Mathematics: A very short introduction.* OUP Oxford, 2002.

"If you go to a research seminar and someone mentions an open problem and you find it interesting, don't just think, "Oh, well this person couldn't solve it, so I certainly couldn't do it." It is often not like that. Give it a go.

If you work on a hard problem, all sorts of benefits can flow from that, even if you don't solve it. One possible benefit applied to me when I worked on the distortion problem, which was eventually solved by other people, by the way. If I hadn't thought very, very hard about the distortion problem, in an ultimately fruitless attempt, I wouldn't have solved the unconditional basic sequence problem– I wouldn't have had the ideas I built up from thinking about the distortion problem, which were crucial for the solution. . . .

Solving a problem is a probabilistic process. One good way of increasing your chances in research is to think quite hard about a lot of problems. If you spend a week of serious effort on 10 different problems, then the chance that at some point some little piece of luck will happen that will enable you to solve one of them is surprisingly high.

So, when you're just starting out, you want to cultivate a general interest in mathematics and a readiness to think about things. If you just say, "My adviser has suggested this problem. I've got to solve this problem and there's nothing else," then you're putting all of your eggs in one basket and your probability of success, while I hope that it is not zero, is smaller than it could be."

– Timothy Gowers