Assam Mathematics Olympiad 2021 Solutions

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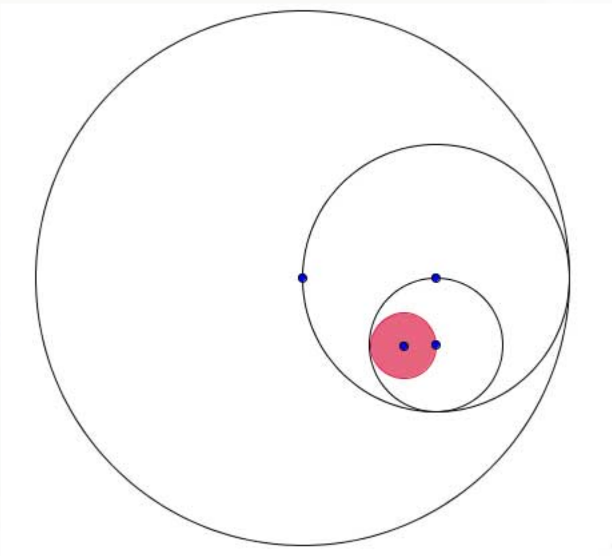
1. Category II

Problem 1. In the given figure, each smaller circle touches the immediate larger circle and passes through its centre. If the area of the largest circle is 4800 sq.cm, what is the area of the shaded circle? 3



Solution. The radius of the shaded circle is $1/8$ of the largest circle. Hence, area $= 4800/64 = 75$. □

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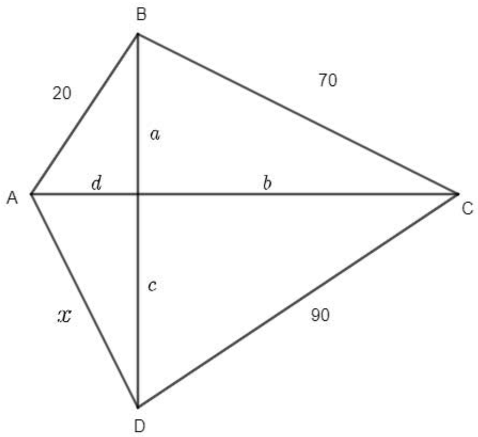
Problem 2. A certain number has exactly 8 factors including 1 and itself. Two of its factors are 21 and 33. Find the sum of the distinct prime factors of this number. 5

Solution. The number is divisible by 3, 7 and 11. Therefore, $1, 3, 7, 11, 21, 33$ are factors of the number. The next factor is 77 and the eighth factor is the number itself. Hence the only prime factors are 3, 7 and 11. The sum of the prime factor is $3 + 7 + 11 = 21$. □

Problem 3. Find the largest integer N for which $N^{80} < 5^{120}$. 5

Solution. Given $N^{80} < 5^{120}$. So, $(N^2)^{40} < (5^3)^{40}$, which gives $N^2 < 5^3 = 125$. Thus, the largest positive integer is 11 as $11^2 < 125$ and $12^2 > 125$. □

Problem 4. ABCD is a convex quadrilateral with perpendicular diagonals. If AB = 20, BC=70 and CD = 90, what is DA? 3



Solution. We label the diagram as shown above. Then, $a^2 + b^2 = 4900$; $b^2 + c^2 = 8100$; $a^2 + d^2 = 400$ and $d^2 + c^2 = x^2$. Adding second and third equations, we get $a^2 + b^2 + c^2 + d^2 = 8500$ which implies $4900 + x^2 = 8500$. So $x^2 = 3600$. Hence $x = 60$. □

Problem 5. How many trailing zeros (i.e. zeros at the end) are there in the product of first 173 natural numbers? 3

Solution. The number of 5's in the product of first 173 natural numbers is $\lfloor \frac{173}{5} \rfloor + \lfloor \frac{173}{25} \rfloor + \lfloor \frac{173}{125} \rfloor = 34 + 6 + 1 = 41$. And there are more than 41 2's. So, there are 41 zeros.[4] □

Problem 6. In how many ways can three distinct numbers a, b, c with $a < b < c$ be choosen from the set $S = \{1, 2, 3, \ldots, 18, 19\}$ so that $a + c = 2b$? 7

[4] $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

Solution. Since $a + c = 2b$, so a and c are both odd or both even. Since $a < b < c$ and the choice is made from $S = \{1, 2, 3, \ldots, 18, 19\}$ and b is AM of a and c, so, choice of a and c guarantees the existence of b in S.

Case I : a & c both odd. Number of choices of a & $c = 10 \times 9 = 90$. But $a < c$ will occur in $\frac{90}{2} = 45$ of the choices.

Case II : a & c are both even. Number of choices of a & $c = 9 \times 8 = 72$. But $a < c$ will occur in exactly half of the choices, i.e. 36.

Thus, total number of ways $= 45 + 36 = 81$. □

Problem 7. Simplify and find the sum of the digits of the number $\sqrt{(44)^2 + (45)^2 + (1980)^2}$. 8

Solution. We have,

$$\begin{aligned}\sqrt{(44)^2 + (45)^2 + (1980)^2} &= \sqrt{(45-44)^2 + 2 \times 45 \times 44 + (1980)^2} \\ &= \sqrt{1 + 2 \times 1980 + (1980)^2} \\ &= \sqrt{(1 + 1980)^2} \\ &= 1981.\end{aligned}$$

Hence, the sum of the digits $= 19$. □

Problem 8. Find the positive square root of abc if

$$\frac{1}{a+3} + \frac{1}{b+3} + \frac{1}{c+3} = \frac{1}{3} \text{ and } a + b + c = 64.$$

6

Solution. Given that

$$\begin{aligned} & \frac{1}{a+3} + \frac{1}{b+3} + \frac{1}{c+3} = \frac{1}{3} \\ \Rightarrow\ & \frac{(b+3)(c+3) + (c+3)(a+3) + (a+3)(b+3)}{(a+3)(b+3)(c+3)} = \frac{1}{3} \\ \Rightarrow\ & \frac{(bc + ca + ab) + 6(a+b+c) + 9}{(a+3)(b+3)(c+3)} = \frac{1}{3} \\ \Rightarrow\ & 3 \times (ab + bc + ca) + 18(a+b+c) + 27 = (a+3)(bc + 3b + 3c + 9) \\ & \qquad = abc + 3(ab + ac + bc) + 9(a+b+c) + 27 \\ \Rightarrow\ & 18 \times 64 = abc + 9 \times 64 \\ \Rightarrow\ & abc = 9 \times 64 \\ \Rightarrow\ & \sqrt{abc} = 24. \end{aligned}$$

□

Problem 9. Let a and b be real numbers such that $a^4 + a^2b^2 + b^4 = 900$ and $a^2 + ab + b^2 = 45$. Find $2ab$. 5

Solution. Given that

$$\begin{aligned}
& a^4 + a^2b^2 + b^4 = 900 \\
\Rightarrow\ & (a^2 + ab + b^2)(a^2 - ab + b^2) = 900 \\
\Rightarrow\ & 45 \times (a^2 - ab + b^2) = 900 \\
\Rightarrow\ & a^2 - ab + b^2 = 20 \\
\Rightarrow\ & 2ab = 45 - 20 \\
\Rightarrow\ & 2ab = 25.
\end{aligned}$$

□

Problem 10. There are $n-1$ red balls, n green balls, and $n+1$ blue balls in a bag. The number of ways of choosing two balls from the bag that have different colors is 299. What is the value of n? 5

Solution. We have,

$$\begin{aligned}
& (n-1)n + n(n+1) + (n+1)(n-1) = 299 \\
\Rightarrow\ & n^2 - n + n^2 + n + n^2 - 1 = 299 \\
\Rightarrow\ & 3n^2 - 1 = 299 \\
\Rightarrow\ & n^2 = 100 \\
\Rightarrow\ & n = 10.
\end{aligned}$$

□

Problem 11. The mean of 11 numbers is 23. If the mean of the first 6 numbers is 21 and that of the last 6 numbers is 26, then find the value of the 6th number? 5

Solution. The sum of the 11 numbers $= 23 \times 11 = 253$. The sum of the first 6 numbers $= 6 \times 21 = 126$. The sum of the last 6 number $= 6 \times 26 = 156$. Total of the first 6 and the last $6 = 282$. Thus,

$$\begin{aligned}
& \text{6th number} + \text{sum of given 11 numbers} = 282 \\
\Rightarrow\ & \text{6th number} = 282 - 253 = 29.
\end{aligned}$$

□

Problem 12. Let $p = 111\ldots11$ (1 appearing $2n$ times) and $q = 777\ldots77$ (7 appearing n times). Find the number of possible values of n such that $p - q$ is a perfect square. 5

Solution. For $n = 1$, $p = 11$ and $q = 7$, so that $p - q = 4$ which is a perfect square.

For $n > 1$, we observe that $p \equiv 11 \pmod{100}$ and $q \equiv 77 \pmod{100}$. Therefore,

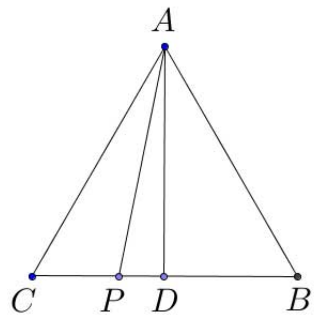
$$\begin{aligned}
& \Rightarrow p - q \equiv -66 \pmod{100} \\
& \Rightarrow p - q \equiv 34 \pmod{100} \\
& \Rightarrow p - q = 100k + 4 \times 8 + 2, \text{ for some } k,
\end{aligned}$$

which is of the form $4K + 2$ and so is not a perfect square. Thus, the number of possible values of n is 1. □

Problem 13. Find the smallest positive integer n such that $n(n + 1)(n + 2)$ is a multiple of 247. 4

Solution. $247 = 13 \times 19$. Multiples of 13 are $13, 26, 39, 52, 65, \ldots$ and multiples of 19 are $19, 38, 57, \ldots$. Observe that 38 and 39 are consecutive. So the smallest such n is 37. □

Problem 14. Let ABC be an equilateral triangle and P be a point on BC. If PB = 50 and PC = 30, then find PA. 5



Solution. We draw AD ⊥ BC. Then, BC = 80, CD = 40, PD = CD − PC = 40 − 30 = 10. The triangle ADP is right angled at D. So, $\mathrm{PA}^2 = \mathrm{PD}^2 + \mathrm{AD}^2 = 10^2 + \left(\frac{\sqrt{3}}{2} \times 80\right)^2 = 4900$. Hence, PA = 70. □

Problem 15. How many 3−digit numbers have sum of the digits equal to 20? 7

Solution. We have $x, y, z \in \{1, 2, 3, \ldots, 9\}$ such that $x + y + z = 20$, i.e. $y + z = 20 - x$. Now, $y + z \leq 18$, so $20 - x \leq 18$, i.e. $x \geq 2$. Similarly, we have $y \geq 2,\ z \geq 2$.

For $x = 2$, $y + z = 18$, we get only one set of values for y, z, i.e. $y = z = 9$.

For $x = 3$, $y + z = 17$, we get two sets of values of y, z corresponding to $y = 9, 8$.

And so on.

Hence, we get $1 + 2 + \ldots + 8 = 36$ such numbers. □

Problem 16. A company mixes two varieties of chemical fertilizers A and B in the ratio 2 : 3. A cost Rs 300 per kg and B cost Rs 250 per kg. The company sells the mixture at Rs 324 per kg. What is the profit or loss percentage? 4

Solution. Per kg of mixture contains $\frac{2}{3}$ kg of A and $\frac{3}{5}$ kg of B.

$$\begin{aligned} \text{Cost Price} &= \frac{2}{5} \times 300 + \frac{3}{5} \times 250 \\ &= 120 + 150 \\ &= 270. \end{aligned}$$

SP $= 324$. So, Profit$= 324 - 270 = 54$. Profit percentage $= \frac{54}{270} \times 100\% = 20\%$. □

Problem 17. If $xy = 6$ and $x^2y + xy^2 + x + y = 63$, find the value of $x^2 + y^2$. 5

Solution. Given that $xy = 6$ and $x^2y + xy^2 + x + y = 63$. Thus,

$$\begin{aligned} & xy(x+y) + (x+y) = 63 \\ \Rightarrow\ & (x+y)(xy+1) = 63 \\ \Rightarrow\ & (x+y)(6+1) = 63 \\ \Rightarrow\ & (x+y) = 9 \\ \Rightarrow\ & x^2 + y^2 = 69. \end{aligned}$$

□

Problem 18. Let p and q be two $3-$digit numbers such that $pq = 888888$. Find the remainder when $p + q$ is divided by 88. 5

Solution. We have,

$$\begin{aligned} pq &= 888888 \\ &= 2^3 \times 3 \times 7 \times 11 \times 13 \times 37. \end{aligned}$$

Expressing pq as product of 3 digit integers, we get

$$\begin{aligned} pq &= 2 \times 2 \times 2 \times 3 \times 7 \times 11 \times 13 \times 37 \\ &= 4 \times 21 \times 11 \times 2 \times 13 \times 37 \\ &= 924 \times 962. \\ \therefore p + q &= 1886. \end{aligned}$$

Hence the remainder when $p + q$ is divided by 88 is 38. □

Problem 19. If $P(x)$ be a polynomial of degree 47 having positive integer coefficients such that the product of the coefficients is 31. What is the remainder when $P(x)$ is divided by $x - 1$? 6

Solution. Since 31 is prime and product of all coefficient is 31, so all coefficients are 1 except one of them which is 31. In any case, remainder when $P(x)$ is divided by $(x - 1)$ is

$$\begin{aligned} P(1) &= 31 + (1 + 1 + \cdots + 1)\ \text{47 times} \\ &= 78. \end{aligned}$$

□

Problem 20. A rectangular field of length 20 m and area 320 sq.m has a uniform path of area 160 sq.m along its entire boundary. Find the width of the path. 4

Solution. Let the width of the path be x. Then,

$$\begin{aligned}&(20+2x)(16+2x)-320=160\\ \Rightarrow\ &x^2+18x-40=0\\ \Rightarrow\ &x=2.\end{aligned}$$

□

2. Category III

Problem 1. Find the face value of the digit x such that the number $3332962592221851481 11x59$ is divisible by 37. 3

Solution. Since $1000 \equiv 1 \pmod{37}$, we get

$$\begin{aligned}333296259222185148111x59 &\equiv 333+296+259+222+185+148+111+100x+59 \pmod{37}\\ &\equiv 100x+59 \pmod{37}\\ &\equiv 26x+22 \pmod{37}\\ &\equiv -11x+22 \pmod{37}\\ &\equiv -11(x-2) \pmod{37},\end{aligned}$$

which will be congruent to $0 \pmod{37}$ when $x=2$. □

Problem 2. There is a question bank of 625 math problems on the topics Algebra, Geometry and Number Theory. You are given the task of preparing a question paper by selecting a few questions randomly from the question bank without looking at them but the question paper should have at least 5 Algebra questions or at least 7 Geometry questions or at least 11 Number Theory questions. What is the smallest number of questions that you should select? 4

Solution. In worst case, it may happen that there are 4 algebra, 6 geometry and 10 number theory. So, $4+6+10=20$. But if we select 21 questions then we can guarantee that at least 5 algebra or at least 7 geometry or at least 11 number theory questions will come. Note that the number 625 has no role to play here except that it is greater than 21. □

Problem 3. Let n be a positive integer. The sum of the divisors of $2n$ is 195, and that of $\frac{n}{2}$ is 39. What is the sum of the divisors of n? 6

Solution. Since $\sigma(\frac{n}{2})$ is a positive integer, therefore n is divisible by 2. Let $n = 2^{\alpha}m$, where $\gcd(2, m) = 1$. Therefore,

$$\begin{aligned}\sigma(2n) &= (2^{\alpha+2} - 1)\sigma(m)\\ &= (2 \times 2^{\alpha+1} - 1)\sigma(m)\\ &= (3 \times 2^{\alpha+1} - 2 \times 2^{\alpha} - 1)\sigma(m)\\ &= \big(3 \times (2^{\alpha+1} - 1) - 2 \times (2^{\alpha} - 1)\big)\sigma(m)\\ &= 3\sigma(n) - 2\sigma(\frac{n}{2}).\end{aligned}$$

Therefore, $\sigma(n) = 91$. □

Problem 4. Find the number formed by the tenth and units digit of the following number 5

$$2021^{2020^{2019}} + 2019^{2020^{2021}}.$$

Solution. We have, $2021 \equiv 21 \pmod{100}$ & $2019 \equiv 19 \pmod{100}$, $\gcd(21, 100) = 1$ & $\gcd(19, 100) = 1$. $\phi(100) = \phi(25 \times 4) = \phi(25)\phi(4) = (5^2 - 5)(2^2 - 2) = 40$. By Fermat's Little Theorem, $21^{\phi(100)} \equiv 1 \pmod{100} \Rightarrow 21^{40} \equiv 1 \pmod{100}$ and $19^{\phi(100)} \equiv 1 \pmod{100} \Rightarrow 19^{40} \equiv 1 \pmod{100}$. Thus, $2021^{40} \equiv 1 \pmod{100}$ and $2019^{40} \equiv 1 \pmod{100}$.

Hence, we have

$$\begin{aligned}2021^{2020^{2019}} &= 2021^{2020^2 \times 2020^{2017}}\\ &= 2021^{40 \times 101 \times 1010 \times 2020^{2017}}\\ &= (2021^{40})^{101 \times 1010 \times 2020^{2017}}\\ &\equiv 1 \pmod{100}\end{aligned}$$

$$\begin{aligned}2019^{2020^{2021}} &= 2019^{2020^2 \times 2020^{2019}}\\ &= 2019^{40 \times 101 \times 1010 \times 2020^{2019}}\\ &= (2019^{40})^{101 \times 1010 \times 2020^{2019}}\\ &\equiv 1 \pmod{100}\end{aligned}$$

Thus, $2021^{2020^{2019}} + 2019^{2020^{2021}} \equiv 2 \pmod{100}$. So the required number is 02 or 2. □

Problem 5. Let a and b be positive integers. What is the greatest positive integer n for which the following inequality can not hold? 6

$$\frac{(a+b)^3}{a^2 b} < n.$$

Solution. We have, $a + b = \frac{a}{2} + \frac{a}{2} + b$. AM-GM inequality gives

$$\begin{aligned} \frac{\frac{a}{2} + \frac{a}{2} + b}{3} &\geq \sqrt[3]{\frac{a}{2}\frac{a}{2}b} \\ \Rightarrow \left(\frac{a+b}{3}\right)^3 &\geq \frac{a^2 b}{4} \\ \Rightarrow \frac{(a+b)^3}{a^2 b} &\geq \frac{27}{4} = 6\frac{3}{4}. \end{aligned}$$

Therefore, the required positive integer is 6. □

Problem 6. Find the remainder when the following is divided by 9. 5

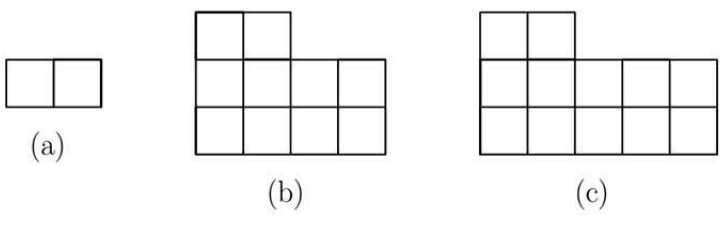
$$4^{50} - 2 \times 4^{49} + 3 \times 4^{48} - 4 \times 4^{47} + \cdots + 47 \times 4^4 - 48 \times 4^3 + 49 \times 4^2 - 50 \times 4 + 51.$$

Solution. Take $-4 = x$. So, the number is

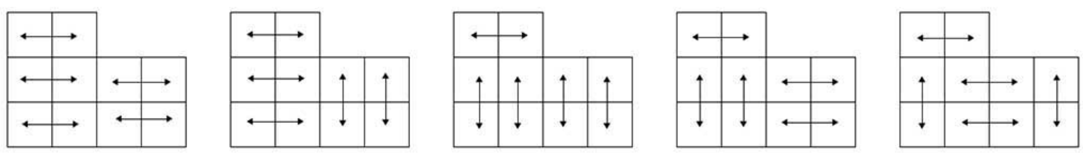
$$\begin{aligned} & x^{50} + 2x^{49} + 3x^{48} + 4x^{47} + 5x^{46} + ... + 47x^4 + 48x^3 + 49x^2 + 50x + 51 \\ &= (x^{50} + 2x^{49} + x^{48}) + 2(x^{48} + 2x^{47} + x^{46}) + 3(x^{46} + 2x^{45} + x^{44}) + \dots \\ &\qquad + 24(x^4 + 2x^3 + x^2) + 25(x^2 + 2x + 1) + 26 \\ &= (x^2 + 2x + 1)(x^{48} + 2x^{46} + 3x^{44} + ... + 24x^2 + 25) + 26 \\ &= 9 \times \text{ an integer} + 26 \\ &\equiv 26 \pmod 9 \equiv 8 \pmod 9. \end{aligned}$$

Therefore the remainder is 8. □

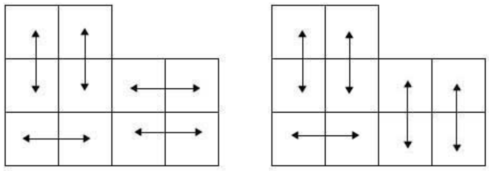
Problem 7. You have got some tiles of the shape shown in figure (a). With those tiles a floor of the shape (b) shown in the figure can be tiled (without any cutting, of course!) in 7 different ways. How many ways can you tile a floor of the shape (c) shown in the figure? 6



Solution. Let us verify for floor (b). If first row tile is place horizontally, then we have the following five possible arrangements:

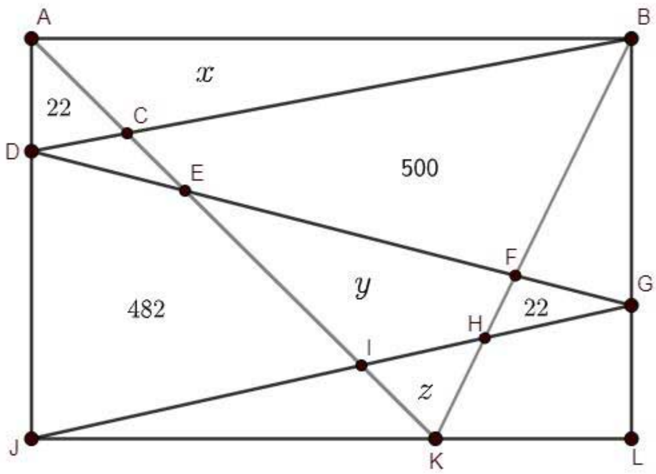


Again, if the first column is given a vertical placement, then the second column also should be vertical. So the third row should be horizontal. Then there are only two possibilities.



Thus there are 7 ways in all. For the shape (c), if the 5th column filled with vertical tile, then the remaining shape is of type (b). So, in this case there are 7 ways of filling up. All other possibilities have 4th and 5th columns filled with horizontal tile. The remaining shape is which can be filled in 4 ways (Try yourself). Thus, the total number of ways $= 7 + 4 = 11$. □

Problem 8. ABLJ is a rectangle. Areas of ACD, BCEF, DEIJ, and FGH are 22, 500, 482, and 22 respectively. Find the area of △HIK. 3



Solution. The areas of the various parts are marked in the figure. Note that,

$$\begin{aligned} & Ar\ \text{ABD} + Ar\ \text{DGJ} = \frac{1}{2}(Ar\ \text{ABLJ}) = Ar\ \triangle\,\text{ABK} \\ \Rightarrow\ & 22 + x + 482 + y + 22 = x + 500 + y + z \\ \Rightarrow\ & z = 504 + 22 - 500 \\ \Rightarrow\ & Ar\,\triangle\,\text{HIK} = 26 \end{aligned}$$

□

Problem 9. What is the greatest integer not exceeding $\left(1 + \dfrac{\sqrt{2}+\sqrt{3}+\sqrt{4}}{\sqrt{2}+\sqrt{3}+\sqrt{6}+\sqrt{8}+4}\right)^{10}$? 3

Solution. We have,

$$\begin{aligned}\sqrt{2}+\sqrt{3}+\sqrt{6}+\sqrt{8}+4 &= \sqrt{2}+\sqrt{3}+2+\sqrt{6}+\sqrt{8}+2\\ &= (\sqrt{2}+\sqrt{3}+\sqrt{4})(1+\sqrt{2})\end{aligned}$$

Therefore,

$$\begin{aligned}\left(1+\frac{\sqrt{2}+\sqrt{3}+\sqrt{4}}{\sqrt{2}+\sqrt{3}+\sqrt{6}+\sqrt{8}+4}\right)^{10} &= \left(1+\frac{1}{1+\sqrt{2}}\right)^{10}\\ &= (\sqrt{2})^{10}\\ &= 32.\end{aligned}$$

□

Problem 10. Let $P(x)$ be a polynomial with positive integer coefficients and $P(0) > 1$. The product of all the coefficients is 73 and the remainder when $P(x)$ is divided by $x-2$ is 2119. What is the degree of $P(x)$? 5

Solution. Since the product of all coefficients is 73, which is prime and the coefficients are positive integers, so all the coefficients must be 1 except one of them which should be 73. Given that $P(0) > 1$. So, the constant coefficient should be 73. Also, the remainder when $P(x)$ is divided by $x-2$ is 2119. So,

$$\begin{aligned}&P(2) = 2119\\ \Rightarrow\ &73+2+2^2+2^3+...+2^n = 2119\\ \Rightarrow\ &73+\frac{2.(2^n-1)}{2-1} = 2119\\ \Rightarrow\ &2.(2^n-1) = 1023\\ \Rightarrow\ &2^n = 1024\\ \Rightarrow\ &n = 10.\end{aligned}$$

□

Problem 11. Let the least values attained by the polynomials x^6+x^3++3 and $x^{14}-3x^7+7$ as x varies over the set of real numbers be a and b respectively. What is the value of $8a+4b$? 4

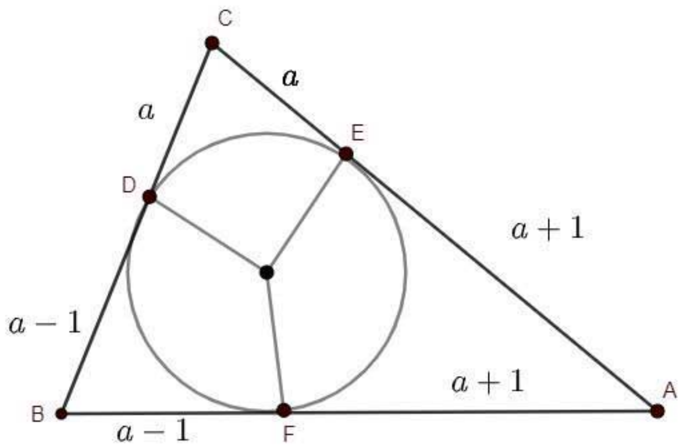
Solution. We have, $x^6+x^3+3 = (x^3)^2+2\times\frac{1}{2}x^3+\frac{1}{4}+3-\frac{1}{4} = (x^3+\frac{1}{2})^2+\frac{11}{4} \geq \frac{11}{4}$. The minimum value attained is $\frac{11}{4}$ as $x^3+\frac{1}{2}=0$ is possible. Thus, $a=\frac{11}{4}$.

Again, $x^{14}-3x^7+7 = (x^7)^2-2\times\frac{3}{2}x^7+\frac{9}{4}+7-\frac{9}{4} = (x^7-\frac{3}{2})^2+\frac{19}{4} \geq \frac{19}{4}$. The minimum value attained is $\frac{19}{4}$ as $x^7-\frac{3}{2}=0$ is possible. Thus, $b=\frac{19}{4}$. So, $8a+4b = 8\times\frac{11}{4}+4\times\frac{19}{4} = 41$. □

Problem 12. Given that a four digit number of the form $xxyy$ is a perfect square. What is its square root? 7

Solution. Let $xxyy = n^2$ so that $11(100x+y) = n^2$. Thus, 11 divides n^2. Hence, 11 must also divide $100x + y$. But 11 divides $99x$. So, 11 divides $100x + y - 99x = x + y$. Since x and y are digits and x is non-zero, so the only possibility is $x + y = 11$. Thus, the options for $xxyy$ are 2299, 3388, 4477, 5566, 6655, 7744, 8833, 9922. The only perfect square out of these is 7744 whose positive square root is 88. □

Problem 13. In triangle ABC, the incircle touches the sides BC, CA, and AB at D, E, and F respectively. If the inradius is 4 units, and BD, CE and AF are consecutive integers, find the perimeter of the triangle ABC. 6

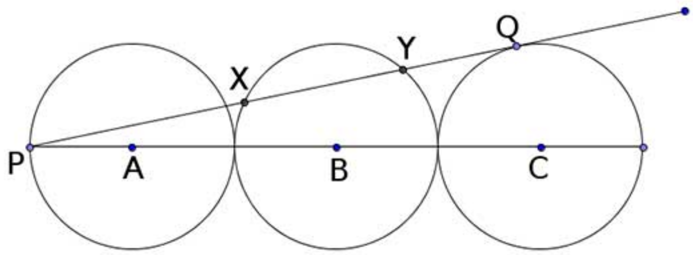


Solution. Since BD, CE and AF are consecutive integers, let $\mathrm{BD} = a-1, \mathrm{CE} = a, \mathrm{AF} = a+1$.

$$\begin{aligned}
& 2S = \mathrm{BC} + \mathrm{CA} + \mathrm{AB} \\
\Rightarrow\ & 2S = 2a - 1 + 2a + 1 + 2a \\
\Rightarrow\ & S = 3a \\
\Rightarrow\ & \sqrt{3a(3a-2a+1)(3a-2a-1)(3a-2a)} = 4.3a \\
\Rightarrow\ & \sqrt{3a(a+1)(a-1)a} = 4.3a \\
\Rightarrow\ & 3a^2(a^2-1) = 4^2.3^2a^2 \\
\Rightarrow\ & a = 7.
\end{aligned}$$

So, perimeter $2S = 6a = 42$. □

Problem 14. Three circles of radius 20 each are arranged such that their respective centres A, B, and C are collinear and the middle circle touches the other two. PQ is a tangent to the third circle and intersects the middle circle at X and Y. Find the length XY. 6



Solution. Let us draw BD ⊥ XY. PQ is tangent at Q. So, CQ ⊥ PQ. So, triangle PBD ∼ triangle PCQ. Hence,

$$\begin{aligned}&\frac{\mathrm{PB}}{\mathrm{PC}}=\frac{\mathrm{BD}}{\mathrm{CQ}}\\ \Rightarrow\ &\frac{60}{100}=\frac{\mathrm{BD}}{\mathrm{CQ}}\\ \Rightarrow\ &\mathrm{BD}=\frac{3}{5}\mathrm{CQ}=\frac{3}{5}\times 20\\ \Rightarrow\ &\mathrm{BY}^2=\mathrm{BD}^2+\mathrm{DY}^2\\ \Rightarrow\ &20^2=12^2+\mathrm{DY}^2\\ \Rightarrow\ &\mathrm{DY}=16\\ \Rightarrow\ &\mathrm{XY}=2\times\mathrm{DY}=32.\end{aligned}$$

□

Problem 15. Let N be the number of integers from 1 to 300 that are not divisible by 3, 5, and 11. What is the largest prime factor of N? 5

Solution. Let A, B, C be the set of integers from 1 to 300 that are divisible by $3, 5, 11$ respectively. $|A| = \left\lfloor \frac{300}{3} \right\rfloor = 100$, $|B| = \left\lfloor \frac{300}{5} \right\rfloor = 60$, $|C| = \left\lfloor \frac{300}{11} \right\rfloor = 27$, $|A \cap B| = \left\lfloor \frac{300}{15} \right\rfloor = 20$, $|B \cap C| = \left\lfloor \frac{300}{55} \right\rfloor = 5$, $|C \cap A| = \left\lfloor \frac{300}{33} \right\rfloor = 9$ and $|A \cap B \cap C| = \left\lfloor \frac{300}{165} \right\rfloor = 1$. Thus,

$$\begin{aligned}|A\cup B\cup C| &= |A|+|B|+|C|-|A\cap B|-|B\cap C|-|C\cap A|+|A\cap B\cap C|\\ &= 100+60+27-20-5-9+1\\ &= 154.\end{aligned}$$

Thus, the number of integers that are not divisible by any of $3, 5, 11$ is $N = 300 - 154 = 146 = 2 \times 73$. The largest prime factor of N is 73. □

Problem 16. A secret lock opens with a few particular ordered triples (x, y, z) of codes, where x, y, z are positive integers with product 5992. How many such codes can open the lock? 6

Solution. We have $5992 = 2^3 \times 7 \times 107$. Since $xyz = 5992$, so x, y, z will be of the forms $x = 2^{a_1} \times 7^{b_1} \times 107^{c_1}$, $y = 2^{a_2} \times 7^{b_2} \times 107^{c_2}$, $z = 2^{a_3} \times 7^{b_3} \times 107^{c_3}$, where $a_1+a_2+a_3 = 3$, $b_1+b_2+b_3 = 1$,

$c_1 + c_2 + c_3 = 1$. Now, the number of non-negative integer solutions of $a_1 + a_2 + a_3 = 3$ is 10. For other two, there are 3 each. Therefore, the number of ordered triples $= 10 \times 3 \times 3 = 90$, which is the required number of such codes. $\square$

Problem 17. If F_r is the number of functions from $\{0, 1\}$ to an r-element set. If $\sum_{r=1}^{n} F_r = 819$, what is the value of n? 5

Solution. The number of functions from $\{0, 1\}$ to an r-element set is r^2. Thus, $F_r = r^2$. Given that $\sum_{r=1}^{n} F_r = 819$. So, we have

$$\begin{aligned} \frac{n(n+1)(2n+1)}{6} &= 819 \\ \Rightarrow n(n+1)(2n+1) &= 6 \times 7 \times 9 \times 13 \\ \Rightarrow n(n+1)(2n+1) &= 13 \times 14 \times 27. \end{aligned}$$

This gives $n = 13$. $\square$

Problem 18. A and B are positive integral multiples of 13 and 17 respectively that add up to 1203. How many possible pairs (A, B) exist? 5

Solution. Let $A = 13x$, $B = 17y$. So, $x, y \geq 0$. So, $13x + 17y = 1203 = 17 \times 70 + 13$, which gives $17(y - 70) = 13(1 - x)$. Therefore 17 divides $1 - x$. Hence $1 - x = 0, -17, -34, -51$, etc. Now, $y - 70 = \frac{13}{17}(1 - x)$ gives $y = 70 + \frac{13}{17}(1 - x)$. So, the values of $1 - x$ that give positive value of y are $0, -17, -34, -51, -68$ and -85. Thus, there are 6 such pairs. $\square$

Problem 19. Let a and b be positive real numbers such that $a\sqrt{a}+b\sqrt{b} = 332$ and $a\sqrt{b}+b\sqrt{a} = 333$. Find $\dfrac{11}{7}(a + b)$. 5

Solution. We have, $a\sqrt{a}+b\sqrt{b} = 332$ and $a\sqrt{b}+b\sqrt{a} = 333$. Adding both we get $(a+b)(\sqrt{a}+\sqrt{b}) = 665$. Let $a = x^2$, $b = y^2$, so that $x^3+y^3 = 332$, and $x^2y+xy^2 = 333$. Hence, $(x+y)^3 = 332+3\times333 = 1331$, so that $x+y = 11$, i.e. $\sqrt{a}+\sqrt{b} = 11$. Therefore, $(a+b)(\sqrt{a}+\sqrt{b}) = 665$ gives $11(a+b) = 665$. Thus, $\frac{11}{7}(a + b) = 95$. $\square$

Problem 20. Let a, b, c, d be distinct integers such that the equation $(x-a)(x-b)(x-c)(x-d) = 25$ has an integral root of r. Find $\dfrac{a + b + c + d}{r}$. 5

Solution. Since r is a root, so $(r - a)(r - b)(r - c)(r - d) = 25$. Again a, b, c, d are distinct, so $r - a, r - b, r - c$ and $r - d$ must be distinct. Also, $(r - a)(r - b)(r - c)(r - d)$ must have factors $\pm1, \pm5$. But, since they are distinct, so exactly two of them will be negative. Again, two of them cannot be -1 and two of them cannot be -5. So the factors will be $1, -1, 5, -5$. In any case, $(r - a) + (r - b) + (r - c) + (r - d) = 0$, which gives $\dfrac{a + b + c + d}{r} = 4$. $\square$