# Primes in subsets of integers

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**Abstract.** I recently completed my PhD in Mathematics from the University of Montreal under the supervision of Prof. Dimitris Koukoulopoulos in August 2021. My thesis [33], 'Primes with a missing digit: distribution in arithmetic progressions and sieve-theoretic applications' is based on the distribution of primes in arithmetic progressions and the existence of infinitely many primes of the form $p = 1 + m^2 + n^2$ with a missing digit. The main techniques used in the thesis are the discrete version of the circle method and the sieve methods. This note gives a short account of the main results on primes with a missing digit in arithmetic progressions from the thesis [33] and the preprint [34] (70 pages long) based on the thesis. In order to motivate our results, we first give a brief account of the prime number theory.

## 1. Prime numbers

One of the fundamental objects of mathematics is the set of natural numbers, that is, the set of counting numbers,

$$1, 2, \ldots .$$

Clearly, there are infinitely many natural numbers, but more importantly, we have the Fundamental theorem of arithmetic: every natural number can be uniquely written as the product of prime numbers. For example, $6 = 2 \times 3$. Here 2 and 3 are the prime numbers. In other words, prime numbers can be thought of as the building blocks for the set of natural numbers. Here we list a few prime numbers:

$$2, 3, 5, 7, 11, 13, 17, \ldots .$$

A natural question arises if there are finitely or infinitely many prime numbers. In this direction, in 300 BC, Euclid gave an elegant proof showing the existence of infinitude of prime numbers by employing the Fundamental theorem of arithmetic. For any real number $x \geq 2$, we let $\pi(x)$ to count the number of primes up to $x$. Then Euclid's result can be expressed mathematically as

$$\lim_{x\to\infty} \pi(x) = \infty.$$

Euclid's result is impressive. However, it does not give us the rate of growth of $\pi(x)$ as $x$ varies. For instance, how many prime numbers are there up to $10^{1000}$. Of course, trivially, we have $\pi(x) \leq x$. So, a genuine question is to determine the true size of the function $\pi(x)$ for some large $x$. The mathematical community had to wait till the end of the 18th century to at least guess what the size of $\pi(x)$ will be. In 1792-93, Carl Friedrich Gauss conjectured that the density of prime numbers around a large real number $x$ is $1/\log x$. In other words,

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log x} = 1.$$

The proof of the above statement, known as the Prime Number Theorem (PNT)[1], turned out to be unexpectedly difficult. The above conjecture reveals that the true size of $\pi(x)$ is 'roughly' $x/\log x$ for a large real number $x$.

In the 1850s, Chebyshev showed that there exists two positive real numbers $C_1, C_2 > 0$ such that

$$C_1 \frac{x}{\log x} \leq \pi(x) \leq C_2 \frac{x}{\log x}. \tag{1.1}$$

Furthermore, Chebyshev showed that if $\lim_{x\to\infty} \pi(x)\log x/x$ exists, then the limit has to be equal to 1. However, he could not show the existence of the limit, which is the crux of the PNT.

To prove the estimate in (1.1), Chebyshev needed to find a quantity that we comprehend independently of what we know about primes, but at the same time it can be expressed in terms of primes. Chebyshev's key idea was that the central binomial coefficient served the purpose. Note that the central binomial coefficient $\binom{2n}{n}$ is an integer for all positive integer $n$, and it is divisible by all primes $p \in (n, 2n]$.

In 1859, Riemann wrote his famous memoir, where he outlined his plan to prove the Prime Number Theorem. The key idea in Riemann's approach was to use complex analysis and the theory of analytic continuation to attack this seemingly naive counting problem of primes. Riemann introduced what is now called the Riemann-zeta function[2]:

$$\zeta(s) := \sum_{n\geq 1} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1},$$

[1] For an excellent introduction to the prime number theory, we ask the interested readers to look at Davenport's book [7]. In particular, most of the results mentioned in Sections 1 and 2 can be found there, and for those topics not covered, an attempt has been made to give proper reference.

[2] Leonard Euler also considered the same zeta function, however, $s > 1$ is restricted to only real numbers.

for[3] $\{s \in \mathbb{C} : \mathrm{Re}(s) > 1\}$. One can then extend the zeta function analytically to the whole complex plane, except for a simple pole at $s = 1$. The Prime Number Theorem (PNT) would then follow if one could show that $\zeta(s) \neq 0$ for $\mathrm{Re}(s) = 1$. This idea was then independently utilized by Hadamard and de la Vallée-Poussin to finally settle the proof of the PNT in 1896.

The proof of the prime number theorem can be considered as one of the greatest triumphs in mathematics of 19th century. The striking feature being, in order to solve a seemingly naive question of prime numbers, we need to use complex function theory. One can ask if there is different proof of the PNT that avoids use of complex analysis. To the surprise of the mathematical community, in 1950, Atle Selberg [36] and Paul Erdős [9] independently proved PNT without using complex analysis (which is sometimes referred to as the *elementary proof of the prime number theorem*[4]).

## *Notation*

We employ some standard notation that will be used throughout this article.

- Expressions of the form $\mathfrak{f}(x) = O(\mathfrak{g}(x))$, $\mathfrak{f}(x) \ll \mathfrak{g}(x)$ and $\mathfrak{g}(x) \gg \mathfrak{f}(x)$ signify that $|\mathfrak{f}(x)| \leq C|\mathfrak{g}(x)|$ for all sufficiently large $x$, where $C > 0$ is an absolute constant. A subscript of the form $\ll_A$ means the implied constant may depend on the parameter $A$. The notation $\mathfrak{f}(x) \asymp \mathfrak{g}(x)$ indicates that $\mathfrak{f}(x) \ll \mathfrak{g}(x) \ll \mathfrak{f}(x)$.
- We write $f(x) \sim g(x)$ to denote $f(x)/g(x) \to 1$ as $x \to \infty$.
- All sums, products and maxima will be taken over $\mathbb{N} = \{1, 2, \dots\}$ unless specified otherwise.
- We reserve the letter $p$ to denote primes.
- For any integers $a$ and $b$, $(a, b)$ will denote its greatest common divisor (gcd).
- For any set $\mathcal{B}$, $\#\mathcal{B}$ denote the cardinality of the set $\mathcal{B}$ and $1_{\mathcal{B}}$ will denote the indicator function for the set $\mathcal{B}$, that is $1_{\mathcal{B}}(x) = 1$ if $x \in \mathcal{B}$ and 0, otherwise.
- For any natural number $n$, $\varphi(n)$ counts the number of positive integers less than $n$ and co-prime to $n$.
- We will denote the set of complex numbers by $\mathbb{C}$.

## 2. Prime numbers in arithmetic progressions

The next step is to investigate the distribution of primes in arithmetic progressions. For example, what can we say about the primes of the form $4n+1$ or $4n+3$. Are there infinitely many primes of the

[3] Here $\mathbb{C}$ denotes the set of complex numbers.

[4] We would like to emphasize that *elementary* does not mean 'easy' in number theory, rather it is widely used in the context of those proofs that avoid the use of complex analysis.

form, $4n+1$ or $4n+3$? In 1837, Dirichlet answered this question with the seminal result in analytic number theory: for any two relatively positive co-prime integers $a$ and $q$, there are infinitely many primes in the progression $\{a+nq : n \text{ non-negative integer}\}$. In order to prove this, he introduced a class of functions now known as *Dirichlet characters*. His idea was to then study, now called a Dirichlet $L$-function,

$$L(s,\chi) = \sum_{n\geq 1} \frac{\chi(n)}{n^s},$$

where $\chi$ is a Dirichlet character. The key and deep input in Dirichlet's proof is that $L(1,\chi) \neq 0$.

Dirichlet showed the existence of infinitely many primes in arithmetic progressions. For $(a,q)=1$, it would be natural to consider the following:

$$\pi(x;q,a) = \#\{p \leq x : p \text{ prime}, p \equiv a \pmod{q}\},$$

that counts the number of primes $\leq x$ and $p \equiv a \pmod{q}$. Heuristically, we expect that for $q \leq x^{1-\epsilon}$, $\epsilon > 0$, we have

$$\pi(x;q,a) \sim \frac{\pi(x)}{\varphi(q)} \quad \text{as} \quad x \to \infty. \tag{2.1}$$

The above relation implies that primes are uniformly distributed in reduced residue classes modulo $q$. However, proving such an estimate in full uniformity in $q$ is a challenging task.

In the 1930s, Walfisz showed that the relation (2.1) holds for $q \leq (\log x)^A$ for some $A > 0$, using a result of Siegel on $L(1,\chi)$. Under the assumption of the Generalized Riemann Hypothesis (GRH), however, the range when (2.1) holds can be extended to $q \leq x^{1/2}/(\log x)^B$ for some $B$ sufficiently large. Moreover, Hugh Montgomery [31] conjectured that the relation (2.1) holds in a much wider range of $q \leq x^{1-\epsilon}$, for some fixed $\epsilon > 0$.

In many applications, however, it suffices to obtain results that only hold on average over $q$. We can therefore substitute the GRH by the celebrated Bombieri-Vinogradov Theorem[5], which was independently established by Enrico Bombieri [1] and A. I. Vinogradov [38] in the 1960s by using the *large sieve*. It states that for any $A > 0$ and $B = B(A)$ sufficiently large in terms of $A$, we have

$$\sum_{q \leq x^{1/2}/(\log x)^B} \max_{(a,q)=1} \left| \pi(x;q,a) - \frac{\pi(x)}{\varphi(q)} \right| \ll_A \frac{x}{(\log x)^A} \tag{2.2}$$

In other words, the Bombieri-Vinogradov Theorem says that "most" $q \leq x^{1/2}/(\log x)^B$ satisfy the relation (2.1) even when the moduli are essentially as large as that can be handled with the GRH.

One of the major open problems in analytic number theory is to improve the exponent $1/2$ in the Bombieri-Vinogradov Theorem. Elliott-Halberstam [8] conjectured that (2.2) should hold for $q \leq x^{1-\epsilon}$ for any fixed $\epsilon > 0$. We note that improvements on the exponent $1/2$ exist for

[5] The Bombieri-Vinogradov theorem has many applications. For instance, the works of Goldston-Pintz-Yildirim, Maynard, Tao on small gaps between primes heavily rest on it. Yitang Zhang also used a variation of the Bombieri-Vinogradov theorem for his celebrated results on bounded gaps between primes.

certain sequences of integers, see, for example, the works of Fouvry [12, 13], Fouvry and Iwaniec [14]. Moreover, Bombieri, Friedlander and Iwaniec [2, 3, 4] wrote a series of papers where they established a variant of (2.2) for $q$ that go beyond $x^{1/2}$. More recently, Maynard [28, 29, 30] improved the previous results of Bombieri, Friedlander and Iwaniec.

## 3. Prime numbers in general sets

The above examples illustrate the distribution of prime numbers in certain subsets of natural numbers. More generally, given any infinite subset $\mathcal{N}$ of natural numbers, we can ask about the existence of prime numbers in this set. In general, it is a difficult question and leads to some famous open problems:

- If $\mathcal{N} = \{p+2 : p \text{ prime}\}$, then it is equivalent to asking if there are finitely or infinitely many primes $p$ such that $p+2$ is also a prime number. This is the famous *twin prime conjecture.*
- If we take $\mathcal{N} = \{n^2+1 : n \text{ integer}\}$, then it boils down to asking the infinitude of the primes of the form $n^2+1$, which is also an open problem till date.
- If $\mathcal{N} = \{2^p-1 : p \text{ prime}\}$, then it is conjectured that there are infinitely many *Mersenne primes* $2^p-1$ with $p$ prime. Note that $2^p-1 = 1+2+2^2+\ldots+2^{p-1}$, which implies that the infinitude of Mersenne primes is analogous to asking if there are infinitely many primes with no 0's in their binary expansion.

We cannot prove the infinitude of the primes in the above sets, however, *sieve methods*[6] can give sharp upper bounds for the number of primes in the set $\mathcal{N} \cap [2, x]$ for any large real number x. We take this opportunity to briefly explain the key idea behind sieve methods. Sieve methods are a set of techniques which have been developed to deal with certain problems related to the distribution of primes. The primary goal of sieve methods is to estimate the quantity

$$S(\mathcal{N}, z) := \#\{n \in \mathcal{N} : p|n \implies p \geq z\},$$

which counts the number of elements in the set $\mathcal{N}$ with no prime factors less than a bound $z$. This is done by using a *smoothed version* of the inclusion-exclusion formula together with the distribution of the set $\mathcal{N}$ in arithmetic progressions. For instance, if $\mathcal{N} = \{p+2 : p \leq x\}$ and $z = (x+2)^{1/2}$, then $S(\mathcal{N}, (x+2)^{1/2})$ counts $p \leq x$ such that $p+2$ is a prime $> (x+2)^{1/2}$.

Note that in general, sieve methods alone cannot establish a positive lower bound for the number of primes in the set, $\mathcal{N} \cap [2, x]$ due to the famous parity phenomenon. But, in some particular cases, sieve methods do give the matching lower bound for the number of primes in the set $\mathcal{N} \cap [2, x]$, which requires more 'arithmetic information'.

[6] For a comprehensive account of sieve methods, we invite the interested readers to go through the masterpiece on sieve methods by Friedlander and Iwaniec [17].

We now make a modest attempt to convince the readers why the primes of the form $p = n^2 + 1$ is interesting. Note that the Dirichlet theorem on primes in arithmetic progressions resolves the prime values of any linear polynomials in one variable. The obvious question to ask is to consider the prime values of higher degree polynomials in one variable. Of course, we need the polynomials to be irreducible, or else it cannot take prime values. The polynomial $n^2 + 1$ is the simplest case of the higher degree polynomial, where we are asking for its prime values. Note that the size of the set $\{n^2 + 1 \leq x\}$ is roughly $\sqrt{x}$, which is quite *sparse*. The sparseness is one of the major obstacles to applying various techniques to detect infinitely many prime values of the polynomial $n^2 + 1$.

However, things are a bit different if we consider the polynomials in two variables. Note that it is easier to determine the prime values of $m^2 + n^2$, as it is known that $m^2 + n^2$ is prime if and only if it is 2 or a prime of the form 1 (mod 4). Therefore, it boils down to Dirichlet's theorem. However, the problem becomes much more interesting and at the same time challenging if one restricts one of the variables $m$ or $n$ in $m^2 + n^2$ to be from some special sets. In this direction, in 1997, Fouvry and Iwaniec [15] showed that there exists infinitely many primes of the form $m^2 + n^2$, where $n$ is also prime. Later, in the year 1998, Friedlander and Iwaniec [16] made a big breakthrough by showing the existence of infinitude of primes of the form $m^2 + n^4$. Recently, in 2017, Heath-Brown and Li [20] extended the work of Friedlander-Iwaniec by showing that there exists infinitely many primes of the form $m^2 + n^4$, where $n$ is also prime. All these sets represent the sparse subset of integers. It is also good to mention another landmark example in this regard. In 2002, Heath-Brown [19] showed the existence of infinitude of primes of the form $m^3 + 2n^3$.

## 4. Prime numbers with a missing digit

Motivated by the above discussion for our quest for primes in a sparse set, we now focus on the distribution of prime numbers with some digital restrictions. In order to set up the problem, let $b \geq 3$ be an integer and fix $a_0 \in \{0, 1, \ldots, b-1\}$. Consider[7]

$$\mathcal{A} := \left\{ \sum_{j \geq 0} n_j b^j : n_j \in \{0, \ldots, b-1\} \setminus \{a_0\} \right\},$$

the set of non-negative integers without the digit $a_0$ in their $b$-adic expansion. For any $k \in \mathbb{N}$, the cardinality of the set $\mathcal{A} \cap [1, b^k)$ is $\approx (b-1)^k$. If we set $X = b^k$, then we see that there are $\approx X^{\zeta}$ elements in $\mathcal{A}$ less than $X$, where

$$\zeta := \frac{\log(b-1)}{\log b} < 1.$$

This reveals that $\mathcal{A}$ is a 'sparse set'. It is often the case that sparseness is one of the obstacles in analytic number theory. However, the set $\mathcal{A}$ admits some interesting structure in the sense that its *Fourier transform* has an explicit description, which is often small. There has been a considerable amount of work (see Dartyge-Mauduit [5, 6], Erdős-Mauduit-Sárközy [10, 11], Konyagin

[7] The set $\mathcal{A}$ denotes the set of integers with a missing digit for the entire article, which we will refer to without further comment.

[21], Maynard [25, 26, 27], Pratt [35]) in this direction by exploiting the Fourier structure of the set $\mathcal{A}$.

It is a natural question to ask if the set $\mathcal{A}$ contains infinitely many primes. We expect the answer to be affirmative. In his celebrated paper [26], Maynard showed that for any $X = b^k$ with $b \geq 10$ and $k \to \infty$, the relation

$$\#\{p < X : p \in \mathcal{A}\} \asymp \frac{X^{\zeta}}{\log X}$$

holds. Moreover, for a large base, say $b \geq 2 \times 10^6$, he [25, 27] established an asymptotic formula. If $b$ is a sufficiently large positive integer, then for any choice of $a_0 \in \{0, \ldots, b-1\}$, we have

$$\#\{p < X : p \in \mathcal{A}\} \sim \frac{\kappa X^{\zeta}}{\log X} \quad \text{as} \quad X \to \infty,$$

where

$$\kappa = \frac{b\big(\varphi(b) - 1_{(a_0,\, b)=1}\big)}{(b-1)\varphi(b)}.$$

We are interested in understanding how the primes of $\mathcal{A}$ are distributed in arithmetic progressions. For $(a,q) = 1$ and $(b,q) = 1$, one expects that

$$\#\{p < X : p \equiv a \pmod q, p \in \mathcal{A}\} \sim \frac{\kappa X^{\zeta}}{\varphi(q) \log X} \quad \text{as} \quad X \to \infty$$

holds uniformly for $d \leq X^{\zeta(1-\epsilon)}$, for any fixed $\epsilon > 0$. This seems to be a difficult question at present. Instead, we aim for a Bombieri-Vinogradov Theorem of the following type:

$$\sum_{\substack{q \leq Q \\ (q,b)=1}} \max_{(a,q)=1} \left| \#\{p < X : p \equiv a \pmod q, p \in \mathcal{A}\} - \frac{1}{\varphi(q)} \#\{p < X : p \in \mathcal{A}\} \right| \ll_{A,b} \frac{X^{\zeta}}{(\log X)^{A}},$$

where $Q \leq X^{1/2-\epsilon}$, for any fixed $\epsilon > 0$, provided that $b$ is large enough in terms of $\epsilon$ (so that $\zeta$ is close enough to 1). Unfortunately, using the current techniques, we are unable to prove that the above estimate holds for $Q \leq X^{1/2-\epsilon}$. However, we can prove a weak result in this direction.

For technical convenience, we will work with the von Mangoldt function $\Lambda$ (recall that $\Lambda(n) = \log p$ if $n = p^m$, and 0 otherwise). For $X = b^k$ with $k \in \mathbb{N}$ and for $(a,q) = (r,b) = 1$, we set

$$\mathcal{E}(X; q, a; b, r) := \sum_{\substack{n < X \\ n \equiv a \pmod q \\ n \equiv r \pmod b}} \Lambda(n) 1_{\mathcal{A}}(n) - \frac{1}{\varphi(q)} \frac{b}{\varphi(b)} \sum_{\substack{n < X \\ n \equiv r \pmod b}} 1_{\mathcal{A}}(n).$$

Note that the condition $n \equiv r \pmod b$ in the above is equivalent to $n$ having $r$ as its last digit in its $b$-adic expansion. We add this condition in order to simplify some technical details.

We now state one of the theorems from [33, 34].

**Theorem 1.** *Let $\delta > 0$ and let $b$ be an integer that is sufficiently large in terms of $\delta$. Let $Q \in [1, X^{1/3-\delta}]$ and let $r \in \mathcal{A} \cap [0, b)$ be an integer such that $(r, b) = 1$. Then for any $A > 0$, we have*

$$\sum_{\substack{q \le Q \\ (q,b)=1}} \max_{(a,q)=1} \left|\mathcal{E}(X; a, q; b, r)\right| \ll_{A,b,\delta} \frac{X^{\zeta}}{(\log X)^A}. \tag{4.1}$$

Next, we can do a little better if we allow our moduli to be the product of two integers. However, the parameter $a$ is now fixed, so we must drop from (4.1) the expression $\max_{(a,q)=1}$. We can further have better result in this direction when we replace the absolute value inside the sum over $q$ by a *well-factorable function.* For instance, see [33, 34, Theorems 2, 3].

The proof of Theorem 1 uses a discrete version of the circle method, as used by Maynard [26, 27] in his work. In particular, the proof relies on the Fourier estimates of primes in arithmetic progressions and the Fourier estimates of the set with missing digits.

## *4.1.* ***Primes of the form*** $p = 1 + m^2 + n^2$

We end by giving an application of Theorem 1: we prove the existence of infinitely many primes of the form $p = 1 + m^2 + n^2$ with a missing digit in a large odd base $b$. Before that, we give a brief history on the problem concerning primes of the form $p = 1 + m^2 + n^2$.

The primes of the form $p = m^2 + n^2 + 1$ are interesting for many reasons. Perhaps, it is one of the simplest non-trivial examples of a 'sparse subset of the primes' consisting of the values of a multivariate polynomial. In fact, an application of the sieve methods [32] shows that for any real number $x$, we have

$$\#\{p \le x : p = 1 + m^2 + n^2, p \text{ prime}\} \ll \frac{x}{(\log x)^{3/2}}.$$

It is also known that there are infinitely many primes of the form $p = m^2 + n^2 + 1$, a result due to Linnik [22], who established it by using his *dispersion method.* Later, a sieve-theoretic proof of this was given by Iwaniec [18], making use of the linear and semi-linear sieves in conjunction with the Bombieri-Vinogradov Theorem. For any large real number $x$, Iwaniec's proof also established the matching lower bound[8]

$$\#\{p \le x : p = 1 + m^2 + n^2, p \text{ prime}\} \gg \frac{x}{(\log x)^{3/2}}.$$

We can now state our second theorem [33, 34, Theorem 4].

**Theorem 2.** *Let $b$ be an odd integer that is sufficiently large, and let*

$$\mathbb{B} = \{n : n = n_1^2 + n_2^2 \ \textit{for some } (n_1, n_2) = 1\}$$

[8] We still do not have an asymptotic formula for the number of primes of the form $p = m^2 + n^2 + 1$ up to any positive real number $x$ unconditionally.

*denote the set of integers that have a primitive representation as the sum of two squares. Let* $r \in \mathcal{A} \cap [0, b)$ *with* $\big(r(r-1), b\big) = 1$*. Then we have*

$$\frac{X^{\zeta}}{(\log X)^{3/2}} \ll_b \sum_{\substack{p<X \\ p \equiv r \pmod b}} 1_{\mathcal{A}}(p) 1_{\mathbb{B}}(p-1) \ll_b \frac{X^{\zeta}}{(\log X)^{3/2}}.$$

The implicit upper bound in Theorem 2 follows from Theorem 1 and a standard upper bound sieve estimate. However, for the lower bound, we need to be more careful and use an argument due to Iwaniec [18] that allows sieving for primes of the form $1 + m^2 + n^2$ using the *level of distribution* slightly less than $X^{1/2}$. Additionally, in order to use the sieve estimates efficiently, we need two technical results similar to Theorem 1, which we establish using ideas from Matomäki [23, 24], Maynard [27] and Teräväinen [37].

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