Problem Section 6

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This section contains unsolved problems, whose solutions we ask from the readers, which we will publish in the subsequent issues. All solutions should preferably be typed in LaTeX and emailed to the editor. If you would like to propose problems for this section then please send your problems (with solutions) to the above mentioned email address, preferably typed in LaTeX. Each problem or solution should be typed on separate sheets. Solutions to problems in this issue must be received by *30 June, 2022*. If a problem is not original, the proposer should inform the editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in Ganit Bikash. Solvers are asked to include references for any non-trivial results they use in their solutions.

Problem 14. *Proposed by B. Sury (Indian Statistical Institute, Bengaluru).*

Let $n > 1$ and let $d_1, \ldots, d_r < n$ be a set of proper divisors (possibly not all) of n. Consider the polynomial

$$f_n(t) = t^n - t - \sum_{i=1}^{r} \frac{t^n - 1}{t^{d_i} - 1}.$$

Show that f_n has no integer roots.

Editor's Note: Problem 12 in Volume 71 should have read 2×2 *integer matrices* instead of $n \times n$ *integer matrices*. We regret the mistake and thank *Suman Dowerah* for bringing it to our notice. In light of this, we only propose one problem in this issue and keep Problem 12 open until 30 June, 2022 for solutions from the readers.

Solutions to Old Problems

We did not receive any solutions from the readers for Problems 10 and 11 published in Volume 70. We give the solutions by the proposers for these problems below. Problems 12 and 13 from Volume 71 are still open for solutions from the readers.

Solution 10. *Solved by the proposer. The solution below is by B. Sury (Indian Statistical Institute, Bengaluru).*

For each $k \geq 1$, consider

$$I(k,r) := \int_0^1 \frac{1}{x_1} \int_0^{x_1} \frac{1}{x_2} \cdots \int_0^{x_{r-1}} x_r^{k-1} dx_r \cdots dx_1.$$

Note that only the last integrand is x_r^{k-1}; others are $\frac{1}{x_i}$. Simplification leads to $I(k,r) = \frac{1}{k^r}$ easily which gives

$$\sum_{k=1}^{n} I(k,r) = \sum_{k=1}^{n} \frac{1}{k^r}.$$

On the other hand, rewrite $I(k,r)$ with the variable x_r changed to $y_r = 1 - x_r$. Summing over all k from 1 to n, we have

$$\sum_k I(k,r) = \int_0^1 \frac{1}{x_1} \cdots \int_{1-x_{r-1}}^1 \frac{1-(1-y_r)^n}{y_r} dy_r \cdots dx_1$$

$$= \sum_{l_r \geq 1} (-1)^{l_r - 1} \binom{n}{l_r} \int_0^1 \frac{1}{x_1} \cdots \int_{1-x_{r-1}}^1 y_r^{l_r - 1} dy_r \cdots dx_1$$

On simplification, this gives the LHS of our assertion; viz.,

$$\sum_{1 \leq i_1 \leq \leq \cdots \leq i_r \leq n} \binom{n}{i_r} \frac{(-1)^{i_r - 1}}{i_1 i_2 \cdots i_r}.$$

Solution 11. *Solved by the proposer. The solution below is by Anupam Saikia (Indian Institute of Technology Guwahati).*

$\#G = 13 \cdot 101$ and $13 \nmid (101-1)$ so G is cyclic (applying Sylow theorems). Since G has no cycle of length 1313 (> 1111), therefore $G = \langle a \rangle$, where a must be generated by a product of some disjoint 13-cycles and 101-cycles and some 1-cycle, say x number of 13-cycles, y number of 101-cycles and z-number of 1-cycles. It is enough to show $z > 0$. Now, $13x + 101y + z = 1111$ and $z = 0$ would imply that $13 \mid 101(11 - y)$, which forces $y = 11$. But then, $13x = 0$ and $x = 0$, which would imply that a has order 101, a contradiction.