5 Problems 1 solution : Catalan's Identity

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Under the title '5 problems 1 solutions', we intend to discuss 5 problems which can be solved using the same concept. We intend to keep the concept basic. In this particular article, the concept we will be using is as follows.

If n is any positive integer, then

$$\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots+\frac{1}{2n-1}-\frac{1}{2n}=\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\ldots+\frac{1}{2n}.$$

This result is known as the Catalan's identity.

Proof: We have,

$$\begin{aligned}
&\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots+\frac{1}{2n-1}-\frac{1}{2n}\\
&\qquad=\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\ldots+\frac{1}{2n-1}+\frac{1}{2n}\right)-2\times\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots+\frac{1}{2n}\right)\\
&\qquad=\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\ldots+\frac{1}{2n-1}+\frac{1}{2n}\right)-\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right)\\
&\qquad=\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\ldots+\frac{1}{2n}.
\end{aligned}$$

To see exactly how this concept will be used, let us look into the following examples.

Problem 1. Prove that for every integer $n > 2$,

$$\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\ldots+\frac{1}{2n}<\frac{5}{6}.$$

(Chile Math Olympiad, Senior, 2019)

Solution: Using Catalan's identity,

$$
\begin{aligned}
&\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\ldots+\frac{1}{2n}\\
&\qquad=\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots+\frac{1}{2n-1}-\frac{1}{2n}\\
&\qquad=\left(\frac{1}{1}-\frac{1}{2}+\frac{1}{3}\right)-\left(\frac{1}{4}-\frac{1}{5}\right)-\left(\frac{1}{6}-\frac{1}{7}\right)-\ldots-\left(\frac{1}{2n-2}-\frac{1}{2n-1}\right)-\frac{1}{2n}\\
&\qquad<\frac{1}{1}-\frac{1}{2}+\frac{1}{3}\\
&\qquad=\frac{5}{6}.
\end{aligned}
$$

Problem 2. If p and q are natural numbers so that

$$\frac{p}{q}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots-\frac{1}{1318}+\frac{1}{1319}.$$

prove that p is divisible by 1979.

(IMO, 1979)

Solution: We have,

$$
\begin{aligned}
\frac{p}{q}&=\left(\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots-\frac{1}{2\times 659}\right)+\frac{1}{1319}\\
&=\left(\frac{1}{660}+\frac{1}{661}+\frac{1}{662}+\ldots+\frac{1}{1318}\right)+\frac{1}{1319}\qquad\text{(Using Catalan's identity)}\\
&=\left(\frac{1}{660}+\frac{1}{1319}\right)+\left(\frac{1}{661}+\frac{1}{1318}\right)+\left(\frac{1}{662}+\frac{1}{1317}\right)+\ldots+\left(\frac{1}{989}+\frac{1}{990}\right)\\
&=\frac{1979}{660\times 1319}+\frac{1979}{661\times 1318}+\frac{1979}{662\times 1317}+\ldots+\frac{1979}{989\times 990}\\
&=1979\times\left(\frac{1}{660\times 1319}+\frac{1}{661\times 1318}+\frac{1}{662\times 1317}+\ldots+\frac{1}{989\times 990}\right).
\end{aligned}
$$

As 1979 is a prime number and all the number 660, 661,…, 1318, 1319 are less than 1979, so p will be a multiple of 1979.

Problem 3. Consider the following 50-term sums:

$$
\begin{aligned}
S&=\frac{1}{1\times 2}+\frac{1}{3\times 4}+\frac{1}{5\times 6}+\ldots+\frac{1}{99\times 100},\\
\text{and } T&=\frac{1}{51\times 100}+\frac{1}{52\times 99}+\frac{1}{53\times 98}+\ldots+\frac{1}{99\times 52}+\frac{1}{100\times 51}.
\end{aligned}
$$

Express $\dfrac{S}{T}$ as an irreducible fraction.

(Argentina, 2014; Pro-RMO, Delhi Region, 2016)

Solution: We have,

$$\begin{aligned}
S &= \frac{1}{1\times 2}+\frac{1}{3\times 4}+\frac{1}{5\times 6}+\ldots+\frac{1}{99\times 100}\\
&= \frac{2-1}{1\times 2}+\frac{4-3}{3\times 4}+\frac{6-5}{5\times 6}+\ldots+\frac{100-99}{99\times 100}\\
&= \frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots+\frac{1}{99}-\frac{1}{100}\\
&= \frac{1}{51}+\frac{1}{52}+\frac{1}{53}+\ldots+\frac{1}{100} \qquad \text{(Using Catalan's identity).}
\end{aligned}$$

Again,

$$\begin{aligned}
T &= \frac{1}{151}\left(\frac{100+51}{51\times 100}+\frac{99+52}{52\times 99}+\frac{98+53}{53\times 98}+\ldots+\frac{52+99}{99\times 52}+\frac{51+100}{100\times 51}\right)\\
&= \frac{1}{151}\left[\left(\frac{1}{51}+\frac{1}{100}\right)+\left(\frac{1}{52}+\frac{1}{99}\right)+\left(\frac{1}{53}+\frac{1}{98}\right)+\ldots+\left(\frac{1}{99}+\frac{1}{52}\right)+\left(\frac{1}{100}+\frac{1}{51}\right)\right]\\
&= \frac{2}{151}\left(\frac{1}{51}+\frac{1}{52}+\frac{1}{53}+\ldots+\frac{1}{100}\right).
\end{aligned}$$

Therefore,

$$\frac{S}{T}=\frac{151}{2}.$$

Problem 4. Prove that

$$\frac{1995}{2}-\frac{1994}{3}+\frac{1993}{4}-\ldots-\frac{2}{1995}+\frac{1}{1996}=\frac{1}{999}+\frac{3}{1000}+\frac{5}{1001}+\ldots+\frac{1995}{1996}.$$

(Baltic Way, 1995)

Solution: We have,

$$\begin{aligned}
\text{LHS} &= \frac{1997-2}{2}-\frac{1997-3}{3}+\frac{1997-4}{4}-\ldots-\frac{1997-1995}{1995}+\frac{1997-1996}{1996}\\
&= 1997\left(\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\ldots-\frac{1}{1995}+\frac{1}{1996}\right)-1\\
&= -1997\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots+\frac{1}{1995}-\frac{1}{1996}\right)+1997-1\\
&= -1997\left(\frac{1}{999}+\frac{1}{1000}+\frac{1}{1001}+\ldots+\frac{1}{1996}\right)+1996\\
&= \left(2-\frac{1997}{999}\right)+\left(2-\frac{1997}{1000}\right)+\left(2-\frac{1997}{1001}\right)+\ldots+\left(2-\frac{1997}{1996}\right)\\
&= \frac{1}{999}+\frac{3}{1000}+\frac{5}{1001}+\ldots+\frac{1995}{1996}.
\end{aligned}$$

Problem 5. Prove that for every natural number $n > 1$,

$$\frac{1}{n+1}\left(\frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2n-1}\right) > \frac{1}{n}\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots+\frac{1}{2n}\right).$$

(RMO, 1998; Canada, 1998, Albania-Ballan TST, 2014)

Solution: We have to prove that

$$n\left(\frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2n-1}\right) > (n+1)\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots+\frac{1}{2n}\right)$$

$$\text{or } n\left(\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\ldots+\frac{1}{2n-1}-\frac{1}{2n}\right) > \frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots+\frac{1}{2n}$$

$$\text{or } n\left(\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\ldots+\frac{1}{2n}\right) > \frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots+\frac{1}{2n}$$

$$\text{or } \frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\ldots+\frac{1}{2n} > \frac{1}{2n}+\frac{1}{4n}+\frac{1}{6n}+\ldots+\frac{1}{2n^2}.$$

As $n+1 < 2n, n+2 < 4n, n+3 < 6n, \ldots, 2n < 2n^2$, we get

$$\frac{1}{n+1} > \frac{1}{2n}, \frac{1}{n+2} > \frac{1}{4n}, \frac{1}{n+3} > \frac{1}{6n}, \ldots, \frac{1}{2n} > \frac{1}{2n^2}.$$

So,

$$\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\ldots+\frac{1}{2n} > \frac{1}{2n}+\frac{1}{4n}+\frac{1}{6n}+\ldots+\frac{1}{2n^2}.$$

Hence proved.

Exercise

Besides the above 5 problems, we give below few more questions on Catalan's identity.

1. Decide whether S_n or T_n is larger, where

$$S_n = \sum_{k=1}^{n} \frac{k}{(2n-2k+1)(2n-k+1)}, \quad T_n = \sum_{k=1}^{n} \frac{1}{k}.$$

(Vietnam, 1983)

2. Prove that

$$\frac{1989}{2} - \frac{1988}{3} + \frac{1987}{4} - \ldots - \frac{2}{1989} + \frac{1}{1990} = \frac{1}{996} + \frac{3}{997} + \frac{5}{998} + \ldots + \frac{1989}{1990}.$$

(Hungary-Israel Binational, 1990)

3. Let p be a prime number greater than 3. Prove that p^2 divides

$$\sum_{i=1}^{k} \binom{p}{i},$$

where $k = \left\lfloor \frac{2p}{3} \right\rfloor$, the greatest integer less than or equal to $\frac{2p}{3}$.

(Putnam, 1996)

4. The number

$$\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots + \frac{1}{2n-1} - \frac{1}{2n}$$

is represented as an irreducible fraction. If $3n+1$ is a prime number, prove that the numerator of this fraction is a multiple of $3n+1$.

(Tournament of towns, Junior A level, 2013)

5. If $p > 2$ is a prime number and m and n are positive integers such that

$$\frac{m}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots - \frac{1}{p-1},$$

then prove that p divides m.

(Bosnia & Herzegovina-Regional, 2011)

$$\frac{1}{2} \times \frac{3}{4} \times \frac{5}{6} \times \ldots \times \frac{99}{100} < \frac{1}{10}.$$

$$i!\overline{i!} = ?$$