Recurrence Relations

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Abstract. In this short note, we will talk about recurrence relations (exploring the combinatorial side). We don't require any prerequisites.

Suppose, we are given a sequence

$$1, 1, 2, 3, 5, 8, 13 \ldots.$$

which satisfies the condition that we can express the next term as the sum of the previous two terms that is $1+1=2, 2+3=5, 3+5=8, 5+8=13,$ and so on . This is an example of a mathematical recurrence.

Linear Recurrence: A sequence $\{a_n\}_{n\geq 0}$ satisfies a **linear recurrence** if a_n is expressed as a linear combination of previous terms of the sequence.

Let's understand it with the above example. For the sequence

$$1, 1, 2, 3, 5, 8, 13 \ldots.$$

We have $a_0 = 1, a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 5, a_5 = 8, a_6 = 13, \ldots.$ And note that, by definition every term satisfies

$$a_n = a_{n-1} + a_{n-2}$$

with $a_0 = 1, a_1 = 1$. So here we are expressing a_n as a linear combination of previous terms (here a_{n-1} and a_{n-2}).

Homogeneous Recurrence Relation: A recurrence relation is **homogeneous** if it doesn't contain any constant terms. Else, it is not homogeneous.

For example, the recurrence $a_n = a_{n-1} + a_{n-2}$ is homogeneous but $a_n = a_{n-1} + a_{n-2} + a_{n-3} + 2$ is not homogeneous. So a linear homogeneous recurrence is of the form

$$c_0 a_n + c_1 a_{n-1} + \cdots + c_r a_{n-r} = 0,$$

where c_i's are integers.

Characteristic equation: The **characteristic equation** of the recurrence is

$$c_0x^r + c_1x^{r-1} + \cdots + c_r = 0$$

If $\alpha_1, \dots \alpha_r$ are the distinct roots then

$$a_n = A_1(\alpha_1)^n + A_2(\alpha_2)^n + \cdots + A_r(\alpha_r)^n, \text{with } A_1, A_2, \dots \text{ constants}$$

What about repetitions? For degree two equations, we have

$$a_n = (A + Bn)r^n, r \text{ is the root, with } A, B \text{ constants.}$$

Enough of theory! Let's try few examples!

Example 1. Solve the recurrence

$$a_n - a_{n-1} - a_{n-2} = 0$$

Proof. Since the recurrence is

$$a_n - a_{n-1} - a_{n-2} = 0.$$

We get that the characteristic equation is

$$x^2 - x - 1 = 0$$

and it's roots are

$$\alpha_1 = \frac{1+\sqrt{5}}{2} \text{ and } \alpha_2 = \frac{1-\sqrt{5}}{2}.$$

Hence

$$a_n = A\Big(\frac{1+\sqrt{5}}{2}\Big)^n + B\Big(\frac{1-\sqrt{5}}{2}\Big)^n.$$

Since

$$a_0 = a_1 = 1 \implies A + B = 1, A\Big(\frac{1+\sqrt{5}}{2}\Big) + B\Big(\frac{1-\sqrt{5}}{2}\Big) = 1.$$

Solving gives us

$$A = \frac{1+\sqrt{5}}{2\sqrt{5}}, B = \frac{-1+\sqrt{5}}{2\sqrt{5}}.$$

So

$$a_n = \frac{1}{\sqrt{5}}\Big[\Big(\frac{1+\sqrt{5}}{2}\Big)^{n+1} - \Big(\frac{1-\sqrt{5}}{2}\Big)^{n+1}\Big].$$

□

Example 2. Find solution to the recurrence relation

$$a_n = 3a_{n-1} + 4a_{n-2}$$

with $a_0 = 2, a_1 = 3$.

Proof. Since the recurrence is

$$a_n = 3a_{n-1} + 4a_{n-2}.$$

We have the charecteristic equation as $x^2 - 3x - 4 = 0 \implies (x-4)(x+1) = 0$. So

$$a_n = A \cdot 4^n + B \cdot (-1)^n.$$

So

$$a_0 = 2 \implies A + B = 2$$

$$a_1 = 3 \implies 4a - B = 3$$

$$\implies A = 1, B = 1.$$

$$\text{Hence } a_n = 4^n + (-1)^n.$$

□

Example 3 (The stamp problem)**.** Suppose we have $1, 2, 5$ valued stamps. The problem is to find the number of ways these can be arranged in a row so that they can add up to a given value n.

Proof. Let a_n be the number of ways the stamp can add up to n. Then we have three cases considering the value of the last stamp.

Case 1: If the last stamp is 1. Then the total value of the remaining stamps must be $n - 1$. Therefore the number of ways in which these remaining stamps can be selected is a_{n-1}.

Case 2: If the last stamp is 2. Then the total value of the remaining stamps must be $n - 2$. Therefore the number of ways in which these remaining stamps can be selected is a_{n-2}.

Case 3: If the last stamp is 5. Then the total value of the remaining stamps must be $n - 5$. Therefore the number of ways in which these remaining stamps can be selected is a_{n-5}.

$$\text{So } \boxed{a_n = a_{n-1} + a_{n-2} + a_{n-5}.}$$

□

The following are practice problems for the reader.

Example 4 (Word with no two consecutive As)**.** Find the number of n letter words using letters from the set $\{A, B\}$ in which no two consecutive A can appear.

Example 5 (Classical Stairs)**.** There is a n stair staircase, one can climb 1 or 2 stairs (1 or 2 steps) at a time, in how many ways he can climb the entire staircase?

Example 6 (2020 C1)**.** Let n be a positive integer. Find the number of permutations $a_1, a_2, \ldots a_n$ of the sequence 1, 2, … , n satisfying

$$a_1 \leq 2a_2 \leq 3a_3 \leq \cdots \leq na_n$$

Hint: Let $F_0 = 1, F_1 = 1, F_2 = 2, F_n = F_{n-1} + F_{n-2}$. *We claim that the number of permutation is* F_n.

Example 7 (2018 USAJMO)**.** For each positive integer n, find the number of n-digit positive integers that satisfy both of the following conditions:no two consecutive digits are equal, and the last digit is a prime.