5 Problems 1 solution : The perfect square method

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Under the title '5 problems 1 solution', we intend to discuss 5 problems which can be solved using the same concept. We intend to keep the concept basic. In this particular article the concept we will be using is as follows.

If 'a' is any real number, then $a^2 \geq 0$, i.e., the square of any real number cannot be negative.

How will this be applied in solving problems?

Suppose, we are given a few real numbers (say a, b, c) with some conditions or equations. On simplifying suppose we get something of this form:

$$(a-3)^2 + (b+2)^2 + (c-5)^2 = 0.$$

But we know that each term of the left hand side is greater than or equal to 0, and so their sum can be zero only if each of them is equal to zero, i.e., $a = 3, b = -2, c = 5$.

To see exactly how this concept will be used, let us look into the following examples.

Problem 1. Find all real number(s) x such that $9^x + 4^x + 1 = 6^x + 3^x + 2^x$.

(Ukraine–1997, Korea–2000)

Solution: Let $2^x = p, 3^x = q$. So, the given equation becomes

$$q^2 + p^2 + 1^2 - pq - q - p = 0,$$

which implies $\frac{1}{2}[(q-p)^2 + (p-1)^2 + (1-q)^2] = 0$. Therefore, $p = q = 1$, i.e., $2^x = 3^x = 1$. Thus $x = 0$ is the only real solution.

Problem 2. Determine the unique pair of real numbers (x, y) that satisfies the equation

$$(4x^2 + 6x + 4)(4y^2 - 12y + 25) = 28.$$

(USA-MTS, 1998-99)

Solution: We have

$$\left[\left(2x+\frac{3}{2}\right)^2+\frac{7}{4}\right]\left[(2y-3)^2+16\right]=\frac{7}{4}\times 16.$$

Therefore, $\left(2x+\frac{3}{2}\right)^2$ and $(2y-3)^2$ must be 0. Thus $x=\frac{-3}{4}$ and $y=\frac{3}{2}$.

Problem 3. Solve the system of equations in real number:

$$(x-1)(y-1)(z-1)=xyz-1,$$
$$\text{and } (x-2)(y-2)(z-2)=xyz-2.$$

(Metropolises, 2018)

Solution: From the first equation we get

$$xy+yz+zx=x+y+z.$$

Again, from the second equation we get

$$(xy+yz+zx)=2(x+y+z)-3.$$

Therefore $x+y+z=3$, and hence $xy+yz+zx=3$. So,

$$\begin{aligned} x^2+y^2+z^2 &= (x+y+z)^2-2(xy+yz+zx)\\ &= 3. \end{aligned}$$

Therefore, $(x^2+y^2+z^2)-(xy+yz+zx)=0$, which implies that

$$(x-y)^2+(y-z)^2+(z-x)^2=0.$$

Hence, $x=y=z=1$.

Problem 4. Three positive real numbers a,b,c are such that $a^2+5b^2+4c^2-4ab-4bc=0$. Can a,b,c be the lengths of the sides of a triangle? Justify your answer.

(RMO, 2014)

Solution: From the given condition we get

$$(a-2b)^2+(b-2c)^2=0,$$

which implies that $a=2b$ and $b=2c$. Therefore $a=4c$. So $a>b+c$. So, a triangle with side lengths a,b,c cannot be formed.

Problem 5. Determine all real pairs (x, y) that satisfy the equations

$$2x^2 + y^2 + 7 = 2(x + 1)(y + 1).$$

(Turkey Junior National Olympiad, 2020)

Solution: From the given equation we get

$$(x - y + 1)^2 + (x - 2)^2 = 0.$$

Therefore, $x - y + 1 = 0$ and $x - 2 = 0$. Hence, $x = 2, y = 3$.

Exercise

1) Find all real numbers x and y such that

$$x^2 + 2y^2 + \frac{1}{2} \leq x(2y + 1).$$

(RMO, 2014)

2) Three real numbers x, y, z are such that

$$x^2 + 6y = -17,$$
$$y^2 + 4z = 1,$$
$$\text{and } z^2 + 2x = 2.$$

What is the value of $x^2 + y^2 + z^2$?

(Pre-RMO, 2013)

3) If the real numbers x, y, z are such $x^2 + 4y^2 + 16z^2 = 48$ and $xy + 4yz + 2zx = 24$, what is the value of $x^2 + y^2 + z^2$?

(Pre-RMO, 2017)

4) Determine all real numbers $x > 1, y > 1$, and $z > 1$ satisfying the equation:

$$x + y + z + \frac{3}{x - 1} + \frac{3}{y - 1} + \frac{3}{z - 1} = 2(\sqrt{x + 2} + \sqrt{y + 2} + \sqrt{z + 2}).$$

(Nordic, 1992)

(Hint: Show that $x + \dfrac{3}{x - 1} - 2\sqrt{x + 2} \geq 0$ for all $x > 1$, etc.)

5) Find all pairs of real numbers (x, y) satisfying

$$(2x + 1)^2 + y^2 + (y - 2x)^2 = \frac{1}{3}.$$

(Croatia, 2003)