# Some Problems on Parity

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## 1. Introduction

Any integer has either odd or even parity. An integer which is exactly divisible by 2 (i.e., leaves remainder 0 when divided by 2) is of even parity and one which is not (i.e., leaves remainder 1 when divided by 2) is of odd parity. It is a very basic idea which can be used to create and solve beautiful problems. We also note the following points:

- Two integers are of same parity if both are even or both are odd.
- Two integers are of opposite parity if one of them is even and other is odd.
- If $a \pm b$ is even, then $a$ and $b$ are of same parity.
- If $a \pm b$ is odd, then $a$ and $b$ are of opposite parity.
- Basic arithmetic operations of parity: odd±odd=even, even±odd=odd, odd×odd=odd, even×odd=even, even×even=even.
- If $\pm a \pm b \pm c$ is even, then at least one of $a$, $b$ and $c$ is even, because if $a$, $b$ and $c$ are all odd, then $\pm a \pm b \pm c$ would be odd.
- Addition or subtraction of any even integer to any other integer does not change it's parity but addition or subtraction of any odd integer to any other integer changes it's parity.

## 2. Problems and Solutions

Below are some problems based on the idea of parity. Some of these problems may have alternative solutions but we are interested in solving them using parity.

## Alternations

**Problem 1:** On a chessboard, a knight starts from square $a1$, and returns there after making several moves. Show that the knight makes an even number of moves. [Mathematical Circles]

**Solution:** At each move, a knight jumps from a square of one colour to a square of opposite colour. Suppose the knight makes an odd number of moves starting from $a1$. In that case the knight will come to a square of opposite colour, which can never be $a1$. So the knight makes an even number of moves if it returns to the starting square $a1$. □

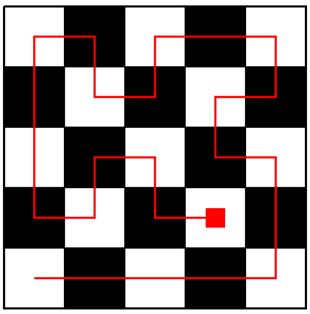
**Problem 2:** Can a knight start at square $a1$ of a chessboard, and go to square $h8$ visiting each of the remaining squares exactly once on the way? [Mathematical Circles]

**Solution:** No. At each move, a knight jumps from a square of one colour to a square of opposite colour. Since the knight must make 63 (odd) moves to visit each of the squares exactly once, so the last move brings him to a square of opposite colour, which can never be $h8$ (since $a1$ and $h8$ are of same colour). □

**Problem 3:** In a $5 \times 5$ chessboard (consider 13 white squares and 12 black squares), a rook covers each and every square of the board exactly once using legal moves. Prove that the square it started is white.

**Solution:** Suppose that the rook started on a black square. Since the rook has to cover 24 (even) squares, so it ends on a black square with 12 steps on white and 13 steps on black squares, which is a contradiction, since there are 12 black squares only. Therefore the rook started on on a white square and also ended on a white square.

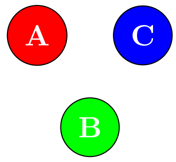
Below is an example of how this task could be accomplished by starting on a white square.



□

**Problem 4:** Three marbles **A**, **B** and **C** are placed on a table as shown. In each move, you can flip one of the marbles to the opposite of the line joining the other two. Can this operation be repeated in any way to attain the original configuration of the marbles after 15 moves?

Original Configuration:




As an example, suppose **B** is flipped to opposite of the line joining **A** and **C**.


**Solution:** No. Observe that: suppose you move from **A** to **B** to **C** in clockwise direction before the operation, then you move from **A** to **B** to **C** in counter-clockwise direction after the operation, and vice-versa (Try this yourself!!). Since the original configuration has the order **A** to **B** to **C** in counter-clockwise direction, so after 15 (odd) moves, the order **A** to **B** to **C** is in clockwise direction, which can never be the original configuration. □

### Odd-even

**Problem 4:** In how many ways can the number 10001 be written as a sum of two primes? [AMC 2011]

**Solution:** Let $a$ and $b$ are primes such that their sum is 10001, i.e., $a+b = 10001$. Since the sum of $a$ and $b$ is odd, so $a$ and $b$ should be of opposite parity, i.e., one of $a$ and $b$ is odd and other is even, but both of them must be prime. The only even prime is 2. Without loss of generality (WLOG), assume that $a$ is odd prime and $b$ is even prime i.e., $b = 2$. Therefore, $a = 10001 - b = 10001 - 2 = 9999$, which is divisible by 9 and hence not a prime. So, 10001 cannot be written as a sum of two primes. Hence, the answer is 0 ways. □

**Problem 5:** Can you find any integer solutions to the equation $5x(x+1) = 3(2y+1)^3$?

**Solution:** No. Clearly, the RHS is odd $\forall\ y \in \mathbb{Z}$. On the other hand, $x(x+1)$ is always even $\forall\ x \in \mathbb{Z}$ because if $x$ is even, then $x+1$ is odd and if $x$ is odd, then $x+1$ is even. This is a contradiction and hence the given equation has no integer solutions. □

**Problem 6:** A function $f : \mathbb{N} \to \mathbb{N}$ is defined by

$$f(x) = \begin{cases} x - 1 \text{ if } x \text{ is even} \\ x + 1 \text{ if } x \text{ is odd} \end{cases}$$

Prove that $f$ is injective.

**Solution:** Let $x_1, x_2 \in \mathbb{N}$ are any two elements. By definition of $f(x)$, $x$ and $f(x)$ have opposite parity i.e., if $x$ is even, then $f(x) = x - 1$ is odd and if $x$ is odd, then $f(x) = x + 1$ is even. So, for $f(x_1) = f(x_2)$, '$x_1$ and $x_2$ are of same parity' is a necessary condition. If $x_1$ and $x_2$ are both even and $f(x_1) = f(x_2)$, then $f(x_1) = f(x_2) \Rightarrow x_1 - 1 = x_2 - 1 \Rightarrow x_1 = x_2$.

If $x_1$ and $x_2$ are both odd and $f(x_1) = f(x_2)$, then $f(x_1) = f(x_2) \Rightarrow x_1 + 1 = x_2 + 1 \Rightarrow x_1 = x_2$. In both the cases, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$. Hence, $f$ is injective. □

**Problem 7:** The product of 50 integers is 1. Can their sum be 0?

**Solution:** No. Clearly, each integer is either $+1$ or $-1$. For the sum to be 0, there should be 25 number of $+1$'s and $-1$'s each. But this is not possible since there must be an even number of $-1$'s so that the product is $+1$. □

**Problem 8:** Let $a_1, a_2, \cdots, a_n$ represent an arbitrary arrangement of the numbers $1, 2, \cdots, n$. Prove that, if $n$ is odd, the product

$$(a_1 - 1)(a_2 - 2) \cdots (a_n - n)$$

is an even number. [Hungary 1906]

**Solution:** Note that the sum $(a_1 - 1) + (a_2 - 2) + \cdots + (a_n - n) = 0$ (since $a_1, a_2, \cdots, a_n$ are $1, 2, \cdots, n$ in some order), an even number. But there are $n$ (odd) factors, so atleast one of the factors is even because if all the factors are odd, then the sum would be odd. Hence, the product is even. □

**Problem 9:** Suppose $a$, $b$ and $c$ are integers such that the equation $ax^2 + bx + c = 0$ has a rational solution. Prove that at least one of the integers $a$, $b$ and $c$ must be even.

**Solution:** Let $\frac{p}{q}$ be the rational solution of the given equation such that $\gcd(p, q) = 1$. Assume, to the contrary, that $a$, $b$ and $c$ are all odd. Plugging $x = \frac{p}{q}$ into the equation $ax^2 + bx + c = 0$ we have $a\left(\dfrac{p}{q}\right)^2 + b\left(\dfrac{p}{q}\right) + c = 0$ or, $ap^2 + bpq + cq^2 = 0$.

Now, since $\gcd(p, q) = 1$ so they cannot be both even. Therefore $p$ and $q$ are either both odd or one of them is even. If $p$ and $q$ are both odd then $ap^2$, $bpq$ and $cq^2$ are also all odd (note that we have assumed that $a$, $b$ and $c$ are all odd) but this is not possible since the sum of three odd numbers is always odd and never equal to 0.

Therefore, one of $p$ and $q$ must be even. WLOG, assume that $p$ is even and $q$. So, $ap^2$ and $bpq$ are even whereas $cq^2$ is odd. But sum of two even numbers and one odd number is an odd number, hence not equal to 0.

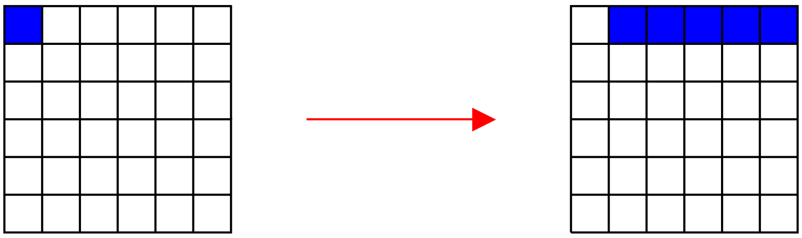
Thus our assumption that $a$, $b$ and $c$ are all odd, leads to a contradiction. Hence, at least one of the integers $a$, $b$ and $c$ must be even. □

**Problem 10:** All natural numbers from 1 to 2021 are written in a row. Can the signs from "+" to "−" be placed between them so that the value of the resulting expression is 0?

**Solution:** No. Observe that $1 + 2 + 3 + \cdots + 2021 = \frac{2021 \cdot 2022}{2} = 2021 \cdot 1011$, which is an odd number. Now we shall prove that changing a "+" to a "−" doesn't change the parity of the sum i.e., the resultant value will always remain odd. Suppose any $i \in \{1, 2, 3, \cdots, 2021\}$ is changed to $-i$, then the value is decreased by $2i$ (since $-i = i - 2i$), an even number and hence it's parity has not changed. Therefore any number we get by writing the natural numbers from 1 to 2021 and placing "+" and "−" signs between them, is odd, and hence not equal to 0. □

**Problem 11:** In a $6 \times 6$ chart, one of the squares is painted blue and remaining white. You are allowed to repaint any column or any row in the chart (i.e., you can select any row or column and flip the colour of all squares within that line). Is it possible to attain an entirely white chart by using only the permitted operations?

Example of Operation:



**Solution:** No. It can be proven that performing a given operation does not change the parity of the number of blue squares i.e., the number of blue squares always stay odd. WLOG, assume that we're flipping all the squares in some row, as the same follows for columns. Suppose that before the operation the row had $x$ blue squares. Since there are 6 rows, so after the operation there will be $6 - x$ blue squares. But note that $x + (6 - x) = 6$, an even number. It follows that $x$ and $6 - x$ are of same parity. This means that the operation has not changed the parity of the number of blue squares i.e., the number of blue squares always stay odd. Hence, it is impossible to attain a completely white chart. □

**Problem 12:** Eight rooks are placed on a chessboard so that none of them attacks another. Prove that the number of rooks standing on black squares is even.

**Solution:** Suppose the squares of chessboard are marked with coordinates such that $(1, 1)$ is in the lower left-hand corner when viewed from white player's side of the chessboard. Then, $(i, j)$ is black if $i + j \equiv 0 \pmod 2$ and white if $i + j \equiv 1 \pmod 2$. Since the rooks do not attack each other, let them be placed on the squares $(1, j_1)$, $(1, j_2)$,$\cdots$, $(1, j_8)$ where $j_1, j_2, \cdots, j_8$ are $1, 2, \cdots, 8$ in some order. Therefore, the parity of the number of rooks on white squares is given by $(1+j_1)+(1+j_2)+\cdots+(1+j_n) \equiv 36+j_1+j_2+\cdots j_8 \equiv 36+36 \equiv 0 \pmod 2$ (note that $j_1, j_2, \cdots, j_8$ are $1, 2, \cdots, 8$ in some order, so their sum is also 36). So the number of rooks on white squares is even and hence the number of rooks standing on black squares is even, since there are 8 rooks in total. □

**Problem 13:** Suppose $r \geq 2$ is an integer, and let $m_1, n_1, m_2, n_2 \cdots m_r, n_r$ be $2r$ integers such that

$$|m_i n_j - m_j n_i| = 1$$

for any two integers $i$ and $j$ satisfying $1 \leq i < j \leq r$. Determine the maximum possible value of $r$. [INMO 2021]

**Solution:** Given equation is $|m_i n_j - m_j n_i| = 1$ or, $m_i n_j - m_j n_i = \pm 1$. Clearly for $r = 2$, $(m_1, n_1) = (1, 0)$ and $(m_2, n_2) = (0, 1)$ satisfies the equation. Also for $r = 3$, $(m_1, n_1) = (1, 0)$, $(m_2, n_2) = (1, 1)$ and $(m_3, n_3) = (0, 1)$ satisfies the equation.
If possible, let $r > 3$. Then we have $m_1 n_2 - m_2 n_1 = \pm 1$, $m_2 n_3 - m_3 n_2 = \pm 1$ and $m_3 n_1 - m_1 n_3 = \pm 1$ (note that the $i < j$ condition is not used here since RHS contains $\pm$). Multiplying the equations by $n_3$, $n_1$ and $n_2$ respectively, we have

$$\begin{cases} m_1 n_2 n_3 - m_2 n_1 n_3 = \pm n_3 \\ m_2 n_3 n_1 - m_3 n_2 n_1 = \pm n_1 \\ m_3 n_1 n_2 - m_1 n_3 n_2 = \pm n_2 \end{cases}$$

Adding, we get $\pm n_1 \pm n_2 \pm n_3 = 0$ which follows that at least one of $n_1$, $n_2$ and $n_3$ is even. WLOG, let $n_1$ be even.
Repeating the same for indices 2, 3 and 4, we have $m_2 n_3 - m_3 n_2 = \pm 1$, $m_3 n_4 - m_4 n_3 = \pm 1$ and $m_4 n_2 - m_2 n_4 = \pm 1$. Multiplying the equations by $n_4$, $n_2$ and $n_3$ respectively, we have

$$\begin{cases} m_2 n_3 n_4 - m_3 n_2 n_4 = \pm n_4 \\ m_3 n_4 n_2 - m_4 n_3 n_2 = \pm n_2 \\ m_4 n_2 n_3 - m_2 n_4 n_3 = \pm n_3 \end{cases}$$

Adding, we get $\pm n_2 \pm n_3 \pm n_4 = 0$ which follows that at least one of $n_2$, $n_3$ and $n_4$ is even. WLOG, let $n_2$ be even.
So we have, both $n_1$ and $n_2$ are even but this implies that $m_1 n_2 - m_2 n_1$ is even, a contradiction. □

## ℑ Pairing up

**Problem 14:** Let $n \geq 2$ be an integer and $T_n$ be the number of non-empty subsets $S$ of $\{1, 2, 3, \cdots n\}$ with the property that the average of the elements of $S$ is an integer. Prove that $T_n - n$ is always even. [Putnam 2006]

**Solution:** This problem can be solved by pairing up the non-empty subsets of $\{1, 2, 3, \cdots n\}$. Clearly, each of the $n$ subsets $\{1\}, \{2\}, \{3\}, \cdots, \{n\}$ has the desired property. Keeping these subsets aside there are $T_n - n$ subsets to be paired up. Here's how the pairing works: Let $S$ be a subset whose average (say, $m$) is an integer.

- If $S$ does not contain $m$, then pair it up with $S \cup \{m\}$.
- If $S$ contains $m$, then pair it up with $S \setminus \{m\}$.

Note that putting the average of a set into the set doesn't change it's average and neither does taking it away. Since we've managed to pair up all the $T_n - n$ subsets, it follows that $T_n - n$ is even. □

**Problem 15:** Alan and Barbara play a game in which they take turns filling entries of an initially empty $2008 \times 2008$ array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled. Alan wins if the determinant of the resulting matrix is non-zero; Barbara wins if it is zero. Which player has a winning strategy? [Putnam 2008]

**Solution:** Here Barbara has the winning strategy. The idea here is to pair up column 1 with column 2.
**Winning strategy of Barbara:**

- If Alan puts an $x$ in the $i$th entry of column 1, then Barbara counters this by putting an $x$ in the $i$th entry of column 2. Conversely, if Alan puts an $x$ in the $i$th entry of column 2, then Barbara counters this by putting an $x$ in the $i$th entry of column 1.
- If Alan writes anything in other columns except 1 and 2, then Barbara also does not write anything in columns 1 and 2.

After the end of the game, columns 1 and 2 are identical and hence the determinant of the matrix is 0. So, Barbara wins the game. □.

**References**

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