Basic concepts of Topology

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1. Introduction

Topology is one of the major mathematical conquests of the twentieth century. It deals with properties that remain unaffected when geometric shapes are bent, twisted, stretched, shrunk or otherwise deformed.

The roots of topology can be traced back to Euler's formula for a particular class of polyhedra which was later extended to a wider class of geometrical shapes by Henri Poincare. Poincaré together with Georg Cantor, Georg Riemann, Mobius and other leading mathematicians of the nineteenth century built up the very foundations of the subject. Topology is so basic in nature that it influences practically every other branch of mathematics. It has found uses in fields like symbolic logic, mechanics and psychology. Because it is not restricted to problems of quantitative nature, it has found applications even in social sciences.

In this brief note we shall focus upon the ideas that motivated the emergence and subsequent development of the subject.

2. Deformation and Homeomorphism

Let us consider a rubber balloon in the shape of a sphere usually denoted by S^2. By stretching the sphere outwards at two opposite points we can transform it into any shape like an ellipsoid or a dumbbell (Figure 1).

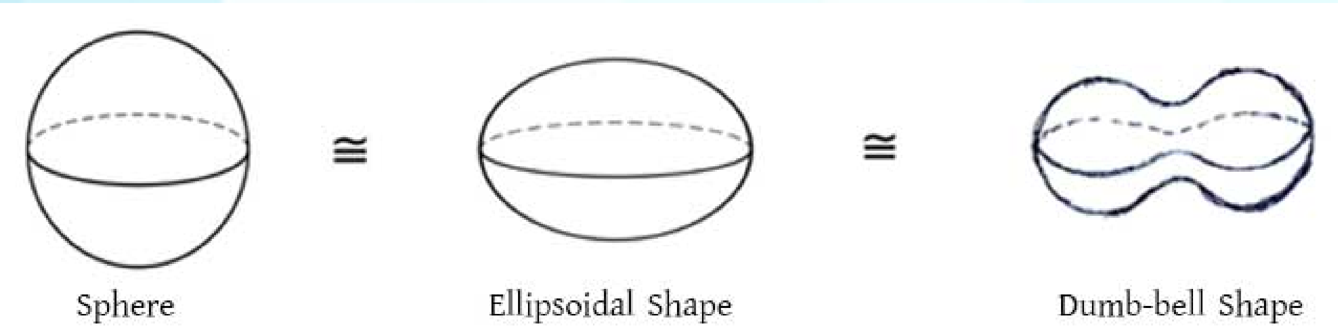

Figure 1

In fact, by stretching or bending or twisting, a sphere may be deformed into an ellipsoidal shape or a dumbbell shape or any other shape in such a way that geometrical properties of the sphere are totally lost in the subsequent deformations. Similarly, an inner-tube or a torus or a doughnut may be deformed into a coffee cup (Figure 2).

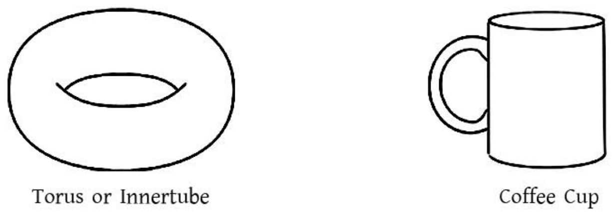

Figure 2

From above it is clear that deformation changes the geometrical properties of a figure.

Though by deformation like stretching, bending or twisting a geometrical shape S_1 may be transformed into a different geometrical shape S_2, there always exists a one-one correspondence between the points of S_1 and S_2 such that for every point P_1 of S_1, we can find a unique point P_2 of S_2 and vice versa.

Figure 3

Mathematically, we can define this correspondence by a bijective function $f : S_1 \to S_2$ so that its inverse $f^{(-1)} : S_2 \to S_1$ is also bijective and both f and $f^{(-1)}$ are continuous. Such a one-one correspondence f between S_1 and S_2 is called a homeomorphism. Therefore, any deformation may be called a homeomorphism in a mathematical sense.

As we have seen, under a homeomorphism (deformation) the geometrical properties of a figure changes. We may now ask here: Is there any property of a geometrical figure that remains invariant under a homeomorphism?

3. Topological Property of a geometrical shape

3.1. Euler's Formula for polyhedra

A polyhedron is a geometrical figure consisting of vertices, edges and faces. Some special types of polyhedra of historical importance are tetrahedron, cube, octahedron, icosahedron and dodecahedron (Figure 4).

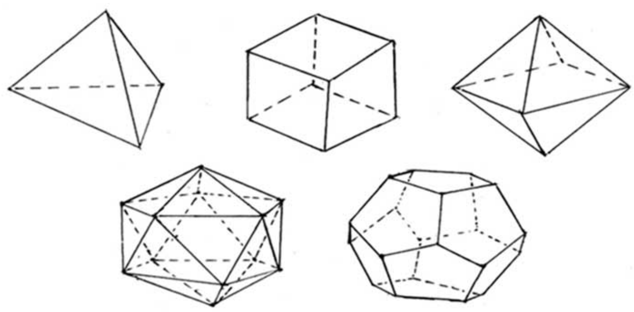

Figure 4

Denoting the number of vertices, edges and faces of a polyhedron by V, E and F respectively, let us find $V-E+F$ which is called the Euler Characteristic of the polyhedron denoted by χ. Thus

$$\chi(P) = V - E + F,$$

where P is a polyhedron.

For a tetrahedron, we have $V = 4, E = 6$ and $F = 4$. Therefore,

$$\chi(\text{tetrahedron}) = V - E + F = 4 - 6 + 4 = 2.$$

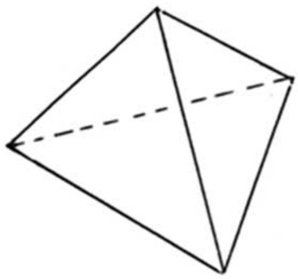

Figure 5

For a cube, we have $V = 8, E = 12, F = 6$. Therefore,

$$\chi(\text{cube}) = V - E + F = 8 - 12 + 6 = 2.$$

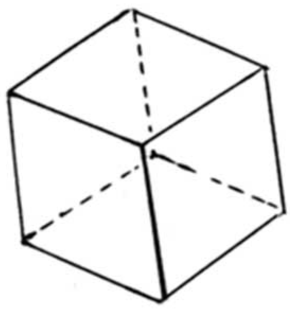

Figure 6

For an octahedron, $V = 6, E = 12, F = 8$. Therefore,

$$\chi(\text{octahedron}) = V - E + F = 6 - 12 + 8 = 2.$$

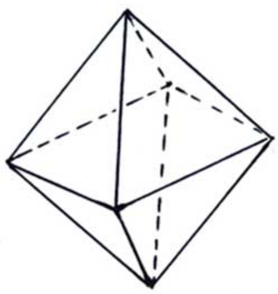

Figure 7

Similarly, we can show that

$$\chi(\text{icosahedron}) = \chi(\text{dodecahedron}) = 2.$$

Therefore, we have the following result.

Theorem: For a polyhedron P, we have,

$$\chi(P) = V - E + F = 2,$$

where P is a tetrahedron or cube or an octahedron or a icosahedron or a dodecahedron. In fact, we can see that Euler's characteristic remains invariant for any polyhedron which is homeomorphic to S^2. Therefore, we have the following result.

Theorem: If a polyhedron P is homeomorphic to a sphere S^2 then

$$\chi(P) = V - E + F = 2.$$

This is known as Euler's formula for a polyhedron P. Thus, Euler's Characteristic χ is a topological property.

3.2. Poincaré's Development of Euler's Characteristics

We may ask ourselves here: What will happen to Euler's Characteristic if a geometrical shape is not homeomorphic to a sphere?

Let us consider a torus, also called a doughnut or an inner-tube. A torus cannot be deformed into a sphere, i.e. a torus is not homeomorphic to a sphere. Let us find the Euler's Characteristic of a torus. We can see that a torus can be deformed into a cube with a hole (Figure 8).

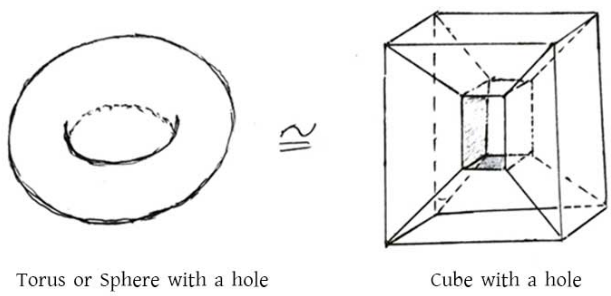

Figure 8

Now we observe that for the cube with a hole $V = 16, E = 32, F = 16$. Thus,

$$\begin{aligned}\chi(\text{cube with one hole}) &= V - E + F \\ &= 16 - 32 + 16 \\ &= 0.\end{aligned}$$

However, cube with a hole may be considered as a polyhedron homeomorphic to a torus which is nothing but a sphere with one hole. Thus, Euler's Characteristic of a polyhedron homeomorphic to a sphere with one hole is 0.

Next consider a sphere with two holes which may be deformed into a cube with two holes (Figure 9).

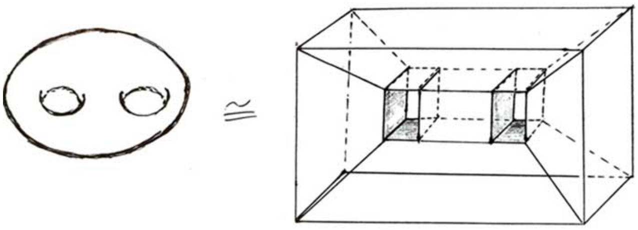

Figure 9

Clearly the cube with two holes is a polyhedron having $V = 24, E = 48, F = 22$. Hence

$\chi(\text{cube with two holes}) = V - E + F = 24 - 48 + 22 = -2.$

This can be stated as follows:

If a polyhedron P is homeomorphic to a sphere with two holes then

$$\chi(P) = -2.$$

Similarly, if a polyhedron P is homeomorphic to a sphere with three holes then

$$\chi(P) = -4.$$

Let us make this observation as follows:

For a polyhedron P,

- $\chi(P) = 2$, if P is homeomorphic to a sphere with no holes.
- $\chi(P) = 0$, if P is homeomorphic to a sphere with one hole.
- $\chi(P) = -2$, if P is homeomorphic to a sphere with two holes.
- $\chi(P) = -4$, if P is homeomorphic to a sphere with three holes.

In general, we can write $\chi(P) = 2 - 2r$, if P is homeomorphic to a sphere with r holes. This is Poincare's development of Euler Characteristic which remains invariant under homeomorphisms and therefore Euler Characteristic is a topological property.

3.3. Connectivity of Regions

Let us consider a region D enclosed by a circle and another region D_1 enclosed between two concentric circles (Figure 10), i.e. a circle with one hole.

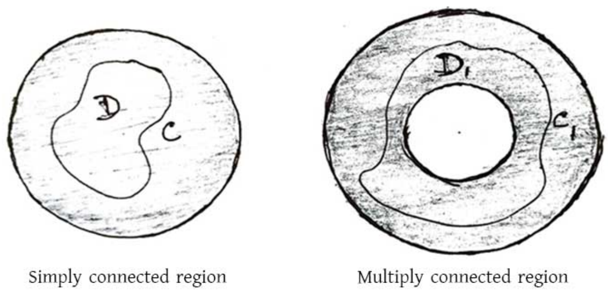

Figure 10

The regions D and D_1 are not homeomorphic.

Now every simple closed curve C in D can be shrunk to a point in D. However, there exists a simple closed curve C_1 in D_1 which cannot be shrunk to a point in D_1 without leaving out of the region D_1. But if we make a cut of the region D_1 from boundary to boundary then it becomes homeomorphic to D and in that case every simple closed curve in D_1 would be shrunk to a point in D_1 (Figure 11).

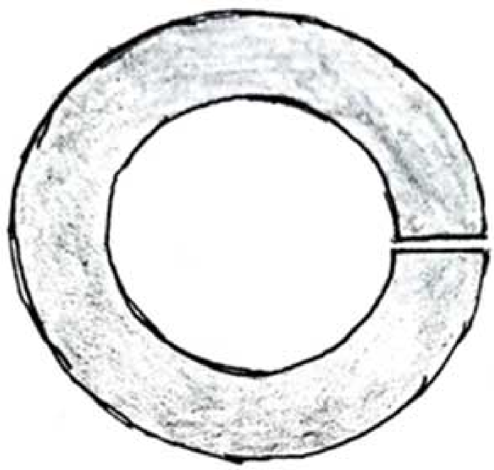

Figure 11

We define a region D to be simply connected, if every simple closed curve can be shrunk to a point in it. A region is called multiply connected if it is not simply connected. With these definitions, we may call a region enclosed by a circle as a simply connected region while a circular region with any number of holes as a multiply connected region. Clearly, the region D_1 in Figure 10 which is multiply connected, can be made simply connected by making one cut across its boundaries (Figure 11). Similarly, a region D_2 enclosed by a circle with two holes is multiply connected. But it can be made simply connected by making two non-intersecting cuts as shown in Figure 12.

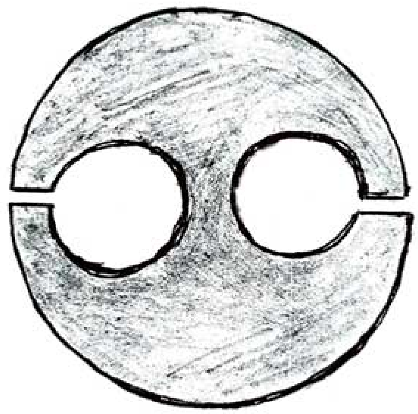

Figure 12

In this way, a region D_n enclosed by a circle with n holes can be made simply connected by making n non intersecting cuts.

In this context we define the degree of connectivity as follows: If a multiply connected region can be made simply connected by making $n-1$ non-intersecting cuts then it is said to have connectivity of degree n. With this terminology, a simply connected region has connectivity of degree 1 since it is already simply connected and $1-1=0$, i.e. no cut is necessary here. A circle with one hole is 2 connected or doubly connected since $2-1=1$ cut is necessary to make it simply connected. A circle with two holes is 3 connected or triply connected since $3-1=2$ cuts are necessary to make it simply connected. It can be proved that the degree of connectivity is a topological property.

4. Conclusion

Our discussion here was simply to motivate on topological properties of surfaces that remain invariant under deformation or homeomorphism. However, in no way it is an exhaustive discussion. We can similarly examine some more properties of surfaces under homeomorphism other than deformation such as genus, knots, Jordan Curve Theorem Fixed Point Theorems etc. For that purpose we have to go a bit deeper into homeomorphisms.