Parity Bias in Integer Partitions

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1. Introduction

A partition of a natural number n is a non increasing sequence of natural numbers called parts that add up to n. For example, the 5 partitions of 4 are 4, 3+1, 2+2, 1+1+1+1, 2+1+1. In this article we will explore the q-series interpretation of partitions and the discovery of a new phenomenon in integer partitions called parity bias in integer partitions. Throughout this article. we use the folowing notations:

$$(a;q)_n = \prod_{k=1}^{n}(1-aq^{k-1})$$

and

$$(a;q)_\infty = \prod_{n=1}^{\infty}(1-aq^{n-1}), \quad |q|<1.$$

The generating function of the partition function is,

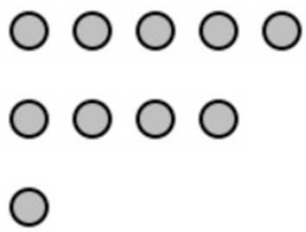
$$P(q) = \sum_{n\geq 0} p(n)q^n = \frac{1}{(q;q)_\infty}, \tag{1.1}$$

where $|q|<1$ throughout the article, unless otherwise mentioned.

Let us now understand the generating function. The right hand side of the above function can be expressed as $\dfrac{1}{(1-q)(1-q^2)(1-q^3)}\cdots$. We know that $\dfrac{1}{1-q}$ $(|q|<1)$ can be written as $1+q+q^2+q^3+\cdots$. So the right hand side of equation (1.1) can be written as $(1+q+q^2+q^3+\ldots)(1+q^2+q^4+q^6+\ldots)(1+q^3+q^6+q^9+\ldots)\cdots$. The first term in the product corresponds to the number of times one appears as a part i.e. q stands for 1 repeating once, q^2 stands for one repeating twice and so on. In the second term, q^2 corresponds to 2 repeating once and likewise we can interpret the rest. The product of all such power series within the brackets, thus gives us the generating function for the number of partitions of a natural number n. For convenience, we take the convention $p(0) = 1$. The power series on the right hand side will expand to $1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + 11q^6 + \ldots$. So, we see that $p(1) = 1, p(2) = 2, p(4) = 5$, etc.

Now it will probably be easy for the readers to understand how the generating functions can be found. For example, the generating function for the number of partitions of a natural number n with all odd parts will be $\frac{1}{(q;q^2)_\infty}$ or the generating function for the number of partitions of a natural number number n with all parts distinct is $(1 + q)(1 + q^2)(1 + q^3)\ldots$, because all the parts can repeat at most once. As an exercise to the readers, I would like them to prove that the number of partitions of a natural number n with all parts distinct is equal to number of partitions of a natural number n with all odd parts. This is a famous result of Euler.

Let us now understand the Ferrers diagram of a partition. The Ferrers diagram is the representation of a partition where we denote a part k by k circles and then place the smaller parts below the larger parts with left alignment. For example, the Ferrer's diagram for 5+4+1 is:



If we replace the circles with boxes, we have the Young's diagram.

The most remarkable work done in this area was probably during the time when Srinivasa Ramanujan and Godfrey Harold Hardy worked together at Cambridge. After them, there have many legendary mathematicians who worked in this area. Prof. Bruce Carl Berndt and Prof. George Eyre Andrews are two notable mathematicians working in this area.

Another beautiful area of research arising from partition theory is the area of overpartitions. An overpartition of n is a non-increasing sequence of natural numbers whose sum is n in which the first occurrence (equivalently, the final occurrence) of a number may be overlined. We denote the number of overpartitions of n by $\overline{p}$(n). Since the overlined parts form a partition into distinct parts and the non-overlined parts form an ordinary partition, we have the generating function:

$$\sum_{n\geq 0} \overline{p}(n)q^n = \prod_{n\geq 1} \frac{1+q^n}{1-q^n} = 1 + 2q + 4q^2 + 8q^3 + 14q^4 + \ldots. \tag{1.2}$$

For example the 8 overpartitions of 3 are $3, \overline{3}, 1 + 1 + 1, \overline{1} + 1 + 1, 2 + 1, \overline{2} + 1, 2 + \overline{1}, \overline{2} + \overline{1}$. It is easy to see that the overlined parts are analogous to the distinct parts of any partition and that is why we have the numerator in the generating function.

2. Parity Bias in Integer Partitions

In 2020, B. Kim. E. Kim and J. Lovejoy [KKL20] discovered a new phenomenon in integer partitions called Parity Bias. It is the situation when the number of odd parts (resp. even) parts are more than the number of even (resp. odd) parts. For example, 3+3+2 is a partition of 8 with more odd parts than even parts and 2+2+2+3+3 is a partition of 12 with more even parts than odd parts.. We now move on to understand the generating functions.

By standard combinatorial arguments, we see that $\frac{q^{bn}}{(q^2;q^2)_n}$ generates partitions into atmost n even parts, exactly n odd parts, and exactly n even parts for $b = 0, 1, 2$ respectively. The argument is as follows: For $b = 0$, consider the representation of an even natural number n with the Ferrer's diagram made of 2's with at most n rows. This can be visualised as follows:

2 2 2 2 2 2
2 2 2 2 2 2
2 2 2 2
2 2 2
2 2 2
2 2 2
2

If there are less than n rows, then we assume the rest of the parts to be 0. Now conjugating this diagram (i.e. interchanging the rows and columns), we get a partition with largest part to be at most n. Similarly, for the case $b = 1$, it is simply adding 1 with each part, and for $b = 2$, it's adding 2 to each part. And since that changes all the 0's to 2's, So, we have exactly n even parts for $b = 2$. Hence $\frac{q^n}{(q^2;q^2)_n}$, and $\frac{q^{2n}}{(q^2;q^2)_n}$ represents the number of partitions with exactly n odd and n even parts respectively.

So, the generating functions for the number of partitions with more odd parts than even parts, $P_o(q)$ and number of partitions with more even parts than odd parts, $P_e(n)$ are:

$$\begin{aligned} P_o(q) &= \sum_{n\geq 0} \frac{q^n}{(q^2;q^2)_n^2} - \sum_{n\geq 0} \frac{q^{3n}}{(q^2;q^2)_n^2}, \\ &= q^3 + q^5 + q^6 + q^7 + \dots, \end{aligned}$$

and

$$\begin{aligned} P_e(q) &= \frac{1}{(q;q)_\infty} - \sum_{n\geq 0} \frac{q^n}{(q^2;q^2)_n^2}, \\ &= q^2 + 2q^4 + 2q^6 + q^7 + \dots. \end{aligned}$$

3. Some results in Parity Bias of integer Partitions

The first result of this new phenomenon of parity bias is the following.

Theorem 1. *(B. Kim, E. Kim, J. Lovejoy). The number of partitions of n ($\neq 2$) with more odd parts than even parts is always more than the number of partitions of n ($\neq 2$) with more even parts than odd parts, i.e. the difference*

$$P_o(q) - P_e(q) = \sum_{n\geq 0} \frac{q^n}{(q^2;q^2)_n^2} - \sum_{n\geq 0} \frac{q^{3n}}{(q^2;q^2)_n^2} - \frac{1}{(q;q)_\infty} + \sum_{n\geq 0} \frac{q^n}{(q^2;q^2)_n^2}$$

has positive coefficients for all powers of q except q^2.

Define $\overline{p}_u(n)$ to be the number of overpartitions of n with more unoverlined parts than overlined parts and $\overline{p}_o(n)$ to be the number of overpartitions of n with more overlined parts than unoverlined parts. The generating function are as follows:

$$\overline{P}_o(q) = \sum_{n\geq 0} \overline{p}_o(q)q^n = \sum_{n\geq 0} \frac{q^{n(n+1)/2}}{(q)_n}\frac{1}{(q)_n} - \sum_{n\geq 0} \frac{q^{n(n+1)/2}}{(q)_n}\frac{q^n}{(q)_n},$$

$$\overline{P}_u(q) = \sum_{n\geq 0} \overline{p}_u(q)q^n = \frac{(-q)_\infty}{(q)_\infty} - \sum_{n\geq 0} \frac{q^{n(n+1)/2}}{(q)_n}\frac{1}{(q)_n}.$$

The reader is asked to convince themself that this is indeed the generating function. The part $\frac{q^{n(n+1)/2}}{(q)_n}$ is for the number if partitions with exactly n overlined parts (Hint: Ferrers Diagram). The following remarkable result was proved by Kim, Kim and Lovejoy.

Theorem 2. *(B. Kim, E. Kim, J. Lovejoy). The difference $\overline{p}_u(n)$ - $\overline{p}_o(n)$ is equal to the number of overpartitions of n where the number of unoverlined parts is at least two more than the number of overlined parts.*

Let $p_{j,k,m}(n)$ be the number of partitions of n such that there are more parts congruent to j modulo m than parts congruent to k modulo m for $m \geq 2$. The next theorem given by B. Kim and E. Kim, generalizes the earlier result.

Theorem 3. *(B. Kim, E. Kim). For all positive integers $n \geq m^2 - m + 1$, $p_{1,0,m}(n) > p_{0,1,m}(n)$, i.e. the difference*

$$\sum_{n\geq 1}(p_{1,0,m} - p_{0,1,m}(n))q^n = \frac{(q^m;q^m)_\infty}{(q;q)_\infty} \sum_{n\geq 1} \frac{q^{mn^2-(m-1)n}(1-q^{(m-1)n})}{(q^m;q^m)_n^w}$$

has positive coefficients for $n \geq m^2 - m + 1$.

The generating functions for the last theorem are;

$$P_{1,0,m}(q) = \sum_{n\geq 0} p_{1,0,m}(n)q^n = \frac{(q;q^m;q^m)_\infty}{(q;q)_\infty} \sum_{n\geq 0} \frac{q^n}{(q^m;q^m)_n^2} - \frac{(q;q^m;q^m)_\infty}{(q;q)_\infty} \sum_{n\geq 0} \frac{q^{(m+1)n}}{(q^m;q^m)_n^2},$$

$$P_{0,1,m}(q) = \sum_{n\geq 0} p_{0,1,m}(n)q^n = \frac{1}{(q;q)_\infty} - \frac{(q;q^m;q^m)_\infty}{(q;q)_\infty} \sum_{n\geq 0} \frac{q^n}{(q^m;q^m)_n^2}.$$

Let us try to understand the above generating functions for the last theorem. In the Ferrers diagram made of 2's, if we replace all the 2's with m's, then we see that $\frac{q^{bn}}{(q^m;q^m)_n}$ is the generating function with exactly n parts congruent to b modulo m for $b \neq 0$ and for $b = 0$, it is the generating function with at most n parts congruent to b modulo m. So, $\sum_{n\geq 0} \frac{q^n}{(q^m;q^m)_n^2} - \sum_{n\geq 0} \frac{q^{(m+1)n}}{(q^m;q^m)_n^2}$ is the number of partitions with more parts congruent to 1 modulo m than parts congruent to 0 modulo m because the first part is the generating function with exactly n parts congruent to 1 modulo m and at most n parts congruent to m modulo n and the second part is the generating function with exactly n parts congruent to 1 modulo m and exactly n parts congruent to m modulo n. $\frac{(q;q^m;q^m)_\infty}{(q;q)_\infty}$ is the generating function for the remaining parts as there is no restriction on the remaining parts. This is how the generating function of $P_{1,0,m}(q)$ is constructed. And for $P_{0,1,m}(q)$ we subtract the first part from the total number of ordinary partitions.

B. Kim and E. Kim also proved the following result:

Theorem 4. *(B. Kim and E. Kim). Let $m \geq 2$ be an integer. Then $p_{1,m,m}(n) \geq p_{m,1,m}(n)$, and $p_{1,m-1,m}(n) \geq p_{m-1,1,m}(n)$.*

Following this result, Shane Chern proved an extension of this result.

Theorem 5. *(Shane Chern). Let $m \geq 2$ be an integer. For any two integers a and b with $1 \leq a < b \leq m$, we have $p_{a,b,m}(n) \geq p_{b,a,m}(n)$.*

4. Concluding Remarks

Parity Bias is an extremely new area of research. There is a lot to be explored in this area. I would say that this is just the tip of the iceberg. For further insights into this area, the readers can look at the references.

References

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