A Bijective Counting of Dyck Paths by Catalan Numbers

Bishal Deb

University College London

E-mail: `bishal@gonitsora.com`

Abstract. In this article, we construct a bijection and its inverse to show that Dyck paths are counted by the Catalan numbers.

1. Introduction

The *Catalan numbers* is one of the most favourite sequences, if not the most favourite sequence, of combinatorialists. The first few Catalan numbers are

$$1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900, 2674440, 9694845, \dots$$

which is the sequence A000108 in the OEIS. The n^{th} Catalan number is given by the famous formula

$$C_n = \frac{1}{n+1}\binom{2n}{n}. \tag{1}$$

The Catalan numbers are named after the French and Belgian mathematician Eugène Catalan (1814-1894). However, this is also a case of *Stigler's law of eponomy*, which states that no scientific discovery is named after its original discoverer. In fact, Euler knew about this sequence and mentions it in a letter that he wrote to Goldbach in 1751. The earliest known reference to this sequence is by the Mongolian mathematician Minggatu (1692-1763) who discovered it and used it in his works in the 1730s. See [1] and references therein for a detailed historical discussion on Catalan numbers.

The Catalan numbers are ubiquitous in counting problems which is one of the primary reasons for its popularity. It occurs in several counting problems, most often in problems involving objects with a recursive structure. Some of the well known Catalan objects are:

(i) Dyck paths,

(ii) Unlabelled rooted binary trees,

(iii) Number of vertices of an associahedron,

(iv) Non-crossing partitions.

Richard Stanley has compiled a list of 66 different objects in [2, Exercise 6.19], and 214 objects in his book [3], all of which are counted by the Catalan numbers. Video lectures by Xavier Viennot on *The Catalan Garden* are available online at [4, Chapter 2].

In this article, we shall only study Dyck paths, and provide a bijective proof that they are counted by the formula in equation (1).

We first introduce Dyck and binary paths, and then state our main theorem in Section 2. We then study the structure of Dyck and binary paths more closely in Section 3. Finally, we construct the forward bijection in Section 4 and the reverse bijection in Section 5 to show that Dyck paths are counted by the formula in equation (1). However, a detailed proof that our construction works is omitted.

2. Definition of Dyck Paths and Statement of Main Theorem

We use $\mathbb{N}$ to denote the set of non-negative integers which is the set $\{0, 1, 2, \ldots\}$. For $i, j \in \mathbb{Z}$, let $[i, j] \coloneqq \{i, i+1, \ldots, j\}$. Thus, the cardinality of $[i, j]$ is $i-j+1$ whenever $i \leq j$ and 0 otherwise. Let $[n] \coloneqq [1, n]$ for $n \in \mathbb{N}$. By an *integer lattice point* (or *lattice point* in short) we shall mean an ordered pair $(x, y) \in \mathbb{Z}^2$ and we shall think of it as a point in the Cartesian plane. A *step* is an ordered pair of lattice points p_1 and p_2 which we denote by $p_1 \to p_2$. If $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$, then $(x_1 - x_2, y_1 - y_2)$ is called the form of the step $p_1 \to p_2$. We shall think of $p_1 \to p_2$ as a directed line segment starting at p_1 and ending at p_2. Figure 1 is an illustration of the step $(0, 0) \to (3, 2)$. A *lattice path* of length n is a sequence of $n + 1$ lattice points $p_0, \ldots, p_n$ such that $p_i \neq p_j$ for any $0 \leq i, j \leq n$, $i \neq j$. Figure 2 is an example of a lattice path of length 6 given by the sequence of points $(0, 0), (2, 1), (3, -1), (0, 2), (-1, 0), (-2, 2), (2, 3)$.

Definition 2.1. A *binary path* of semilength n is a lattice path of length $2n$ beginning at the origin $(0, 0)$ and ending at $(2n, 0)$ where each step is either of the form $(1, 1)$ or $(1, -1)$. Let $\mathcal{B}_n$ denote the set of all binary paths of semilength n. Figures 3 and 4 are examples of binary paths of semilength 3.

Definition 2.2. A *Dyck path* of semilength n is a binary path of semilength n, in which none of the points are below the x-axis. We use $\mathcal{D}_n$ to denote the set of all Dyck paths of semilength n. Figure 4 is an example of a Dyck path of semilength 3, while Figure 3 is not an example of Dyck path as the path crosses the x-axis.

We shall refer to steps of the form $(1, 1)$ as *northeast steps* and denote such a step using $\nearrow$ (northeast

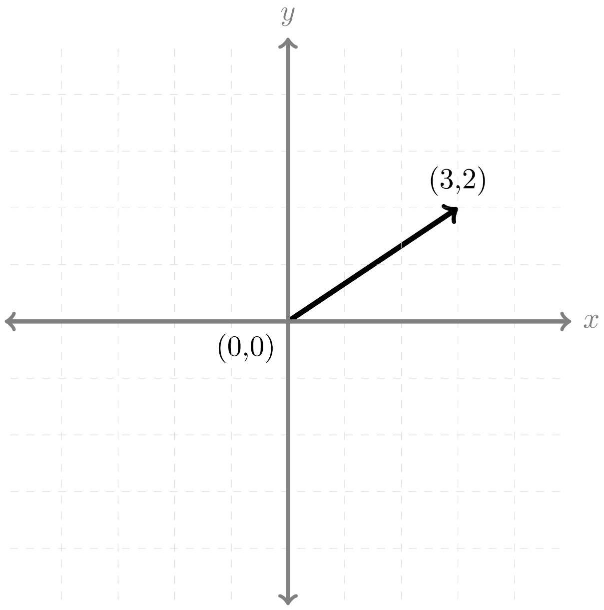


Figure 1: An illustration of the step $(0,0) \to (3,2)$.

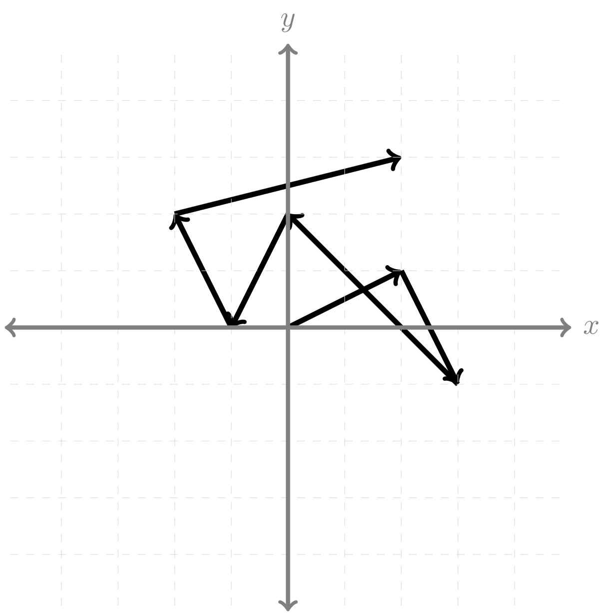


Figure 2: The lattice path of length 6 given by the sequence of points $(0,0),(2,1),(3,-1),(0,2),(-1,0),(-2,2),(2,3)$.

arrow). Similarly, we shall refer to steps of the form $(1,-1)$ as *southeast steps* and denote such a step using $\searrow$ (southeast arrow). We leave it as an exercise to the reader to show that a binary path of semilength n bijectively corresponds to a word[1] of length $2n$ on the alphabet $\{\nearrow, \searrow\}$ with both $\nearrow$ and $\searrow$ occurring exactly n times. We shall call this word the *arrow word* of the binary path.

[1] Given a set A (the alphabet) a word is a finite sequence of elements from A, which are not necessarily distinct. For brevity, we denote the word $(a_1, a_2, \ldots, a_n)$ by $a_1 a_2 \cdots a_n$.

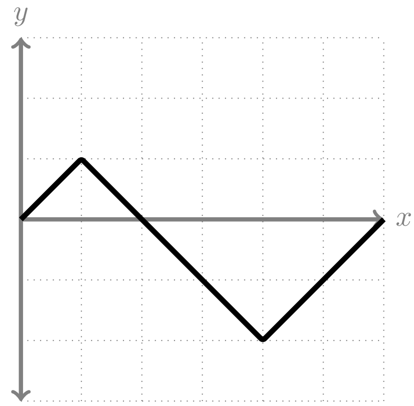


Figure 3: An example of a binary path of semilength 3.

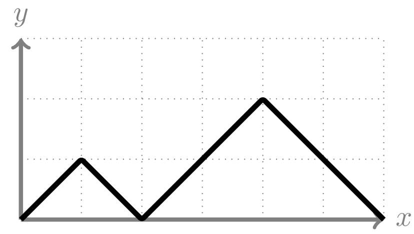


Figure 4: An example of a Dyck path of semilength 3.

For example, arrow word of the binary path in Figure 3 is $\nearrow\searrow\searrow\searrow\nearrow\nearrow$ and the arrow word of the binary path in Figure 4 is $\nearrow\searrow\nearrow\nearrow\searrow\searrow$.

As the number of words of length $2n$ on the alphabet $\{\nearrow, \searrow\}$ with both $\nearrow$ and $\searrow$ occurring exactly n times is $\binom{2n}{n}$, so is the cardinality of $\mathcal{B}_n$.

We intend to show that the cardinality of $\mathcal{D}_n$ is

$$|\mathcal{D}_n| = \frac{1}{n+1}\binom{2n}{n}, \tag{2}$$

We can rewrite this equation as

$$(n+1)|\mathcal{D}_n| = \binom{2n}{n},$$

which is equivalent to showing that

$$|[0,n] \times \mathcal{D}_n| = |\mathcal{B}_n| \tag{3}$$

as we just saw that the cardinality of $\mathcal{B}_n$ is $\binom{2n}{n}$, and we know that the cardinality of $[0,n]$ is $n+1$.

We now state our main theorem:

Theorem 2.1. *The sets $[0,n] \times \mathcal{D}_n$ and $\mathcal{B}_n$ are in bijective correspondence with each other.*

Thus, proving Theorem 2.1 also provides a proof for equations (2) and (3).

3. Structure of Dyck and binary paths

Let $B \in \mathcal{B}_n$ be a binary path given by the sequence of points $p_0, \ldots, p_{2n}$ where $p_i = (x_i, y_i)$ with $p_0 = (x_0, y_0) = (0, 0)$ and $p_{2n} = (x_{2n}, y_{2n}) = (2n, 0)$. Let $w = w_1 \cdots w_{2n}$ be its arrow word. Let $\mathrm{NE}(B) = \{u_1, \ldots, u_n\}$ where $u_1 < \ldots < u_n$ are all the indices for which $w_{u_i} = \nearrow$. Similarly, let $\mathrm{SE}(B) = \{d_1, \ldots, d_n\}$ where $d_1 < \ldots < d_n$ are all the indices for which $w_{d_i} = \searrow$.

For example, if B is the binary path in Figure 3, then $\mathrm{NE}(B) = \{1, 5, 6\}$ and $\mathrm{SE}(B) = \{2, 3, 4\}$. If B is the binary path in Figure 4 then $\mathrm{NE}(B) = \{1, 3, 4\}$ and $\mathrm{SE}(B) = \{2, 5, 6\}$.

The index u_i corresponds to the step

$$(x_{u_i-1}, y_{u_i-1}) = p_{u_i-1} \to p_{u_i} = (x_{u_i}, y_{u_i})$$

where

$$x_{u_i} = x_{u_i-1} + 1 \tag{4}$$

and

$$y_{u_i} = y_{u_i-1} + 1. \tag{5}$$

Similarly, the index d_i corresponds to the step

$$(x_{d_i-1}, y_{d_i-1}) = p_{d_i-1} \to p_{d_i} = (x_{d_i}, y_{d_i})$$

where

$$x_{d_i} = x_{d_i-1} + 1 \tag{6}$$

and

$$y_{d_i} = y_{d_i-1} - 1. \tag{7}$$

In the special case when B is a Dyck path, we have that $y_i \geq 0$ for all $i \in [2n]$ and hence from equation (5), $y_{u_i} \geq 1$ for all indices u_i. It is not difficult to show that for every u_i with $y_{u_i-1} = h$, i.e. the step $p_{u_i-1} \to p_{u_i}$ starts at height h and ends at height $h+1$, there is atleast one $d_j > u_i$ such that $y_{d_j} = h$, i.e. the step $p_{d_j-1} \to p_{d_j}$ starts at height $h+1$ and ends at height h. We use $d_{\sigma(i)}$ to denote the index of the smallest such step, i.e., for $i \in [n]$ with $y_{u_i-1} = h$, $\sigma(i)$ denotes the smallest $j \in [n]$ for which $d_{\sigma(j)} > u_i$ and $y_{d_j} = h$.

Notice that for any $p_k = (x_k, y_k)$ with $u_i < k < d_{\sigma(i)}$, we have that $y_k > h$. Given a northeast step $p_{u_i-1} \to p_{u_i}$, we shall call $p_{d_{\sigma(i)}-1} \to p_{d_{\sigma(i)}}$ its corresponding southeast step. See Figure 5 for a colour coded example.

4. The Forward Bijection $\phi : [0, n] \times \mathcal{D}_n \to \mathcal{B}_n$

Let $D \in \mathcal{D}_n$ be a Dyck path given by the sequence of points $p_0, \ldots, p_{2n}$ where $p_i = (x_i, y_i)$ with $p_0 = (x_0, y_0) = (0, 0)$ and $p_{2n} = (x_{2n}, y_{2n}) = (2n, 0)$. Let $w = w_1 \cdots w_{2n}$ be its arrow word. Let $\mathrm{NE}(D) = \{u_1, \ldots, u_n\}$ where $u_1 < \ldots < u_n$ and let $\mathrm{SE}(D) = \{d_1, \ldots, d_n\}$ where $d_1 < \ldots < d_n$.

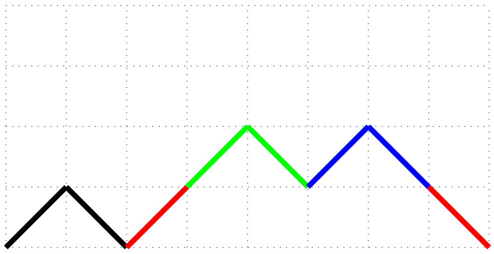


Figure 5: Dyck path for the arrow word $\nearrow\searrow\nearrow\nearrow\searrow\nearrow\searrow\searrow$. The different steps of the form $\nearrow$ and their corresponding steps of the form $\searrow$ have been coloured using the same colour. Here, $u_1 = 1, u_2 = 3, u_3 = 4, u_4 = 6$, $d_1 = 2, d_2 = 5, d_3 = 7, d_4 = 8$ and $\sigma(1) = 1$, $\sigma(2) = 4$, $\sigma(3) = 2$, $\sigma(4) = 3$.

For a fixed $i \in [n]$, let $u = u_i$, $d = d_{\sigma(i)}$, and $h = y_u$. Here h is the height at which the i^{th} northeast step ends. Let $\mathrm{NE}_i(D) \coloneqq \{v_1, \ldots, v_h = u\} \subseteq \mathrm{NE}(D)$ with $v_1 < \ldots < v_h$, where $v_j \leq u$ is the largest index with $y_{v_j} = j$. Let $\mathrm{SE}_i(D) \subseteq \mathrm{SE}(D)$ be the set of indices of the southeast steps corresponding to the steps at the indices $\mathrm{NE}_i(D)$.

Let $a = a_1 \cdots a_{2n}$ be the arrow word where each a_j is defined as follows:

$$a_j = \begin{cases} w_j & \text{if } j \notin \mathrm{NE}_i(D) \text{ and } j \notin \mathrm{SE}_i(D), \\ \searrow & \text{if } j \in \mathrm{NE}_i(D), \\ \nearrow & \text{if } j \in \mathrm{SE}_i(D). \end{cases} \tag{8}$$

We can finally define our map $\phi : [0, n] \times \mathcal{D}_n \to \mathcal{B}_n$. For $(i, D) \in [0, n] \times \mathcal{D}_n$, we define $\phi(i, D)$ as

$$\phi(i, D) = \begin{cases} D & \text{if } i = 0, \\ B & \text{if } i \neq 0\ , \end{cases} \tag{9}$$

where B is the binary path obtained from the arrow word a defined in equation (8).

Let us illustrate this by the example in Figure 6. Let D be the Dyck path of semilength 5 in Figure 6a and let $i = 2$. Then, $\mathrm{NE}_i(D) = \{1, 2\}$ and $\mathrm{SE}_i(D) = \{5, 8\}$ and the steps with these indices have been coloured in red. The arrow word of D is $w = \nearrow\nearrow\ \nearrow\searrow\ \searrow\ \nearrow\searrow\ \searrow\ \nearrow\searrow$, and the arrow word a as per equation (8) is $a = \searrow\searrow\ \nearrow\searrow\ \nearrow\ \nearrow\searrow\ \nearrow\ \nearrow\searrow$. The corresponding binary path $\phi(2, D)$ is the binary path in Figure 6b.

5. The Reverse Bijection $\psi : \mathcal{B}_n \to [0, n] \times \mathcal{D}_n$

Let $B \in \mathcal{B}_n$ be a binary path given by the sequence of points $p_0, \ldots, p_{2n}$ where $p_i = (x_i, y_i)$ with $p_0 = (x_0, y_0) = (0, 0)$ and $p_{2n} = (x_{2n}, y_{2n}) = (2n, 0)$. Let $w = w_1 \cdots w_{2n}$ be its arrow word.

Let $-h = \min_{i \in [0,2n]} y_i$ be the lowest possible height of a point. We know that $h = 0$ if and only if B is a Dyck path.

If $h>0$, then for $1\leq j\leq h$, let d'_j denote the smallest index such that $y_{d'_j}=-j$, and let u'_j denote the largest index such that $y_{u'_j-1}=-j$. Note that the step $p_{d'_j-1}\to p_{d'_j}$ is a southeast step, and that the step $p_{u'_j-1}\to p_{u'_j}$ is a northeast step.

We state the following lemma before constructing our map ψ.

Lemma 5.1. *Let*

$$w=w^{(1)}\searrow w^{(2)}\nearrow w^{(3)}$$

where

$$w^{(1)}=w_1\cdots w_{d'_h-1},$$
$$w^{(2)}=w_{d'_h+1}\cdots w_{u'_h-1},$$
$$w^{(3)}=w_{u'_h+1}\cdots w_{2n}.$$

Also, let w' be the word

$$w'=w^{(1)}\nearrow w^{(2)}\searrow w^{(3)},$$

B' be the binary path given by the arrow word w', and let $p'_0=(x'_0,y'_0),\ldots,p'_{2n}=(x'_{2n},y'_{2n})$ be the points of B'. Then the lowest possible height of B' is

$$\min_{j\in[0,2n]} y'_j=-(h-1).$$

Proof. It is clear that the abscissa of p'_j are given by

$$x'_j=j,$$

and the ordinates are given by

$$y'_j=\begin{cases} y_j & \text{if } j\leq d'_h-1,\\ y_j & \text{if } j\geq u'_h+1,\\ y_j+2 & \text{if } d'_h\leq j\leq u'_h.\end{cases}$$

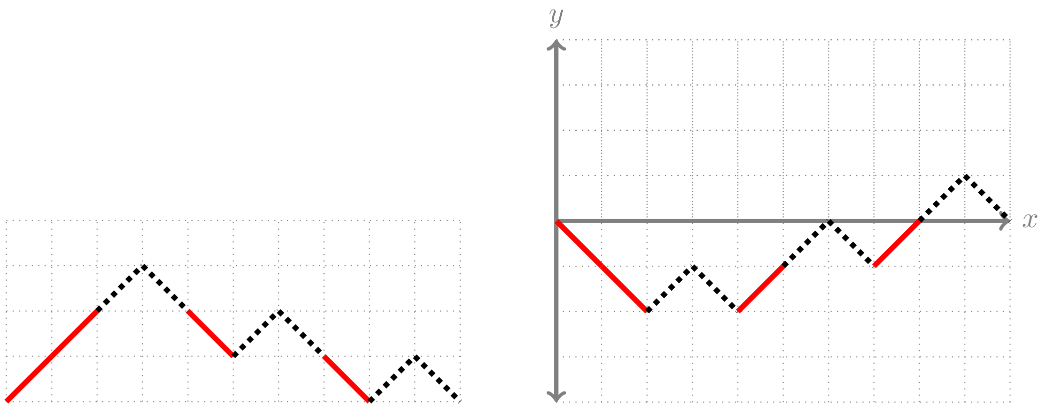
If $j\leq d'_h-1$ or $j\geq u'_h+1$, then the minimum height is $-(h-1)$, and if $d'_h\leq j\leq u'_h$, the minimum height is $-h+2$. Thus, we get that

$$\min_{j\in[0,2n]} y'_j=-(h-1).$$

□

Let $b=b_1\cdots b_{2n}$ be the arrow word where each letter b_i are given by:

$$b_i=\begin{cases} w_i & \text{if } i\neq d'_j \text{ and } i\neq u'_j \text{ for all } 1\leq j\leq h,\\ \searrow & \text{if } i=u'_j \text{ for some } 1\leq j\leq h,\\ \nearrow & \text{if } i=d'_j \text{ for some } 1\leq j\leq h.\end{cases} \tag{10}$$



(a) Arrow word $w = \nearrow\nearrow \nearrow\searrow \searrow \nearrow\searrow \searrow \nearrow\searrow$

(b) Arrow word $w = \searrow\searrow \nearrow\searrow \nearrow \nearrow\searrow \nearrow \nearrow\searrow$

Figure 6: Figure 6a is a Dyck path D of semilength 5 with the arrow word $w = \nearrow\nearrow \nearrow\searrow \searrow \nearrow\searrow \searrow \nearrow\searrow$. For $i = 2$ we have $\mathrm{NE}_i(D) = \{1, 2\}$ and $\mathrm{SE}_i(D) = \{5, 8\}$ and the steps with these indices have been coloured in red.
The binary path B in Figure 6b is of semilength 5 and has the arrow word $w = \searrow\searrow \nearrow\searrow \nearrow \nearrow\searrow \nearrow \nearrow\searrow$. The first and last steps ending at negative heights have been coloured in red.

Let D be the binary path given by the word b. We can see that the word b can be written as

$$b = \underbrace{(\dots((w')')\dots)'}_{h \text{ times}},$$

and hence from Lemma 5.1, D is a Dyck path.

We can finally define the map $\psi : \mathcal{B}_n \to [0, n] \times \mathcal{D}_n$. For $B \in \mathcal{B}_n$, $\psi(B)$ is given by

$$\psi(B) = \begin{cases} (0, B) & \text{if B is a Dyck path,} \\ (i, D) & \text{if B is not a Dyck path} \end{cases} \tag{11}$$

where D is the Dyck path obtained from the arrow word b defined in equation (10), and i is the number of occurences of $\nearrow$ in the prefix $b_1 \cdots b_{d'_h}$ of the word b.

Let us illustrate this by the example in Figure 6. Let B be the binary path of semilength 5 in Figure 6b. Here $h = 2$. The first and the last steps ending at each negative height have been coloured in red. The arrow word of B is $w = \searrow\searrow \nearrow\searrow \nearrow \nearrow\searrow \nearrow \nearrow\searrow$, and the arrow word b as per equation (10) is $b = \nearrow\nearrow \nearrow\searrow \searrow \nearrow\searrow \searrow \nearrow\searrow$. The corresponding pair $\psi(B)$ is $(2, D)$ where D is the Dyck path in Figure 6a.

6. Acknowledgements

I would like to thank one of my gurus, Xavier Viennot, for introducing me to the joys of bijective combinatorics. I constructed this bijection as a solution to an exercise while attending his course

The Art of Bijective Combinatorics Part II ([5]), which he offered at the Institute of Mathematical Sciences (Chennai) in 2017.

I would also like to thank the editors of Ganit Bikash for inviting me to write this article and for bearing with my several missed deadlines.

References

[1] Igor Pak (2014), *History of Catalan Numbers*, arXiv: 1408.5711, URL: https://www.math.ucla.edu/p̃ak/papers/cathist4.pdf.

[2] Richard P. Stanley (1999), *Enumerative Combinatorics, Vol. 2*, Cambridge University Press.

[3] Richard P. Stanley (2015), *Catalan Numbers*, Cambridge University Press.

[4] Xavier Viennot (2016), *The Art of Bijective Combinatorics Part I*, Institute of Mathematical Sciences, URL: https://www.viennot.org/abjc1-ch2.html.

[5] Xavier Viennot (2017), *The Art of Bijective Combinatorics Part II*, Institute of Mathematical Sciences, URL: https://www.viennot.org/abjc2.html.

"Unless you are exceptionally brilliant and can solve a long-standing problem of great interest (consider Yitang Zhang coming out of nowhere to make a spectacular breakthrough in number theory), it will be really beneficial to your career to produce your own research problems, the more the better (within reason). Always keep your eyes and ears open to possible interesting problems. If for instance a seminar speaker mentions a problem that you like and is more-or-less in an area about which you are knowledgeable, then don't hesitate to think about it! Don't think, "I never worked on hyperconvex residuated posets, so how could I get anywhere?" Play around with it a little– maybe you will think of something. It might suggest a related question. Do some experiments, gather some data, etc. Doing some computations might suggest a further idea, even if the computations themselves don't seem helpful. Moreover, if you decide to stop working on a problem, do not think that you are giving up. You never know when some random remark at a seminar or in a paper might be the key to further progress. Keep these unsuccessful attempts in the back of your mind, ready to be let out if the door is opened a crack."

– Richard P. Stanley