Problem Section 5

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This section contains unsolved problems, whose solutions we ask from the readers, which we will publish in the subsequent issues. All solutions should preferably be typed in LaTeX and emailed to the editor. If you would like to propose problems for this section then please send your problems (with solutions) to the above mentioned email address, preferably typed in LaTeX. Each problem or solution should be typed on separate sheets. Solutions to problems in this issue must be received by *20 March, 2022*. If a problem is not original, the proposer should inform the editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in Ganit Bikash. Solvers are asked to include references for any non-trivial results they use in their solutions.

Problem 12. *Proposed by B. Sury (Indian Statistical Institute, Bengaluru).*

Let P be a 2×2 integer matrix such that its trace is 0. Show that there exist real $n \times n$ integer matrices A, B such that $P = AB - BA$.

Problem 13. *Proposed by B. Sury (Indian Statistical Institute, Bengaluru).*

Let $p(x) = \sum_{i=1}^{r} a_i x^i, q(x) = \sum_{i=1}^{s} b_i x^i$ be polynomials with integer coefficients. Suppose a positive integer n divides every coefficient of the polynomial $p(x)q(x)$. Prove that n divides $a_i b_j$ for each i, j.

Solutions to Old Problems

We received correct solutions to Problems 7 and 9 from **Amit Kumar Basistha** (Anundoram Borooah Academy, Pathsala, India) and to Problem 8 from **Preyarnsi Saikia** (Indian Institute of Technology Delhi). Problems 10 and 11 from the previous issue are still open.

Solution 7. *Solved by Amit Kumar Basistha and the proposers. The solution below is by Ayan Nath (Chennai Mathematical Institute).*

The only answers are $x \mapsto cx^3$ where c is some real constant, it is easily verified that they work. Make the substitution $x = -a^3 + b^3 + c^3$, $y = a^3 - b^3 + c^3$ and $z = a^3 + b^3 - c^3$, we get

$$f(x) + f(y) + f(z) + 24f\left(\tfrac{1}{2}\sqrt[3]{(x+y)(y+z)(z+x)}\right) = f(x+y+z).$$

Call the above assertion $P(x, y, z)$. Note that $P(0, 0, 0) \implies f(0) = 0$. And $P(-x, x, x) \implies f(-x) = -f(x)$. Observe that

$$P(x, y, 0) \implies f(x) + f(y) + 24f\left(\tfrac{1}{2}\sqrt[3]{xy(x+y)}\right) = f(x+y),$$

$$P(x, y+z, 0) \implies f(x) + f(y+z) + 24f\left(\tfrac{1}{2}\sqrt[3]{x(y+z)(x+y+z)}\right) = f(x+y+z),$$

which implies

$$f(x) + f(y) + f(z) + 24f\left(\tfrac{1}{2}\sqrt[3]{yz(y+z)}\right) + 24f\left(\tfrac{1}{2}\sqrt[3]{x(y+z)(x+y+z)}\right) = f(x+y+z).$$

Therefore

$$f\left(\tfrac{1}{2}\sqrt[3]{yz(y+z)}\right) + f\left(\tfrac{1}{2}\sqrt[3]{x(y+z)(x+y+z)}\right) = f\left(\tfrac{1}{2}\sqrt[3]{(x+y)(y+z)(z+x)}\right).$$

Define $g : \mathbb{R} \to \mathbb{R}$ as $g(x) = f(\frac{1}{2}\sqrt[3]{x})$. The above implies that

$$g\left(yz(y+z)\right) + g\left(x(y+z)(x+y+z)\right) = g\left((x+y)(y+z)(z+x)\right).$$

Since $yz(y+z) + x(y+z)(x+y+z) = (x+y)(y+z)(z+x)$ we hope that we can obtain Cauchy's Functional Equation. Consider the two equations

$$\begin{cases} a &= yz(y+z), \\ b &= x(y+z)(x+y+z). \end{cases}$$

We claim that the above is always solvable in reals for any $a, b \in \mathbb{R}$. Set $y + z = t$. Solving the second equation in x we obtain $x = \dfrac{-t^2 \pm \sqrt{t^4 + 4bt}}{2t}$ and solving the first equation in y we get $y = \dfrac{t^2 \pm \sqrt{t^4 - 4ta}}{2t}$. We just need to ensure that both discriminants are non-negative which is obvious as we take a large t. Therefore we obtain that $g(a) + g(b) = g(a+b)$ for all $a, b \in \mathbb{R}$. Since

g is monotonic it follows that $g(x) \equiv kx$ for some constant $k \in \mathbb{R}$ which implies that $f(x) = 8kx^3$. And we are done.

Remark: The general solution without the monotonicity constraint is given by $f(x) = A(x^3)$ where $A : \mathbb{R} \to \mathbb{R}$ is any additive function.

The solution by Amit Kumar Basistha was also along similar lines. There were two more solutions submitted by the proposers.

Editor's Note: This problem appeared in an obscure AoPS olympiad, Universal Mathematical Olympiad 2020 as Problem 1.

Solution 8. *Solved by Preyarnsi Saikia and the proposer. The solution below is by Preyarnsi Saikia (Indian Institute of Technology Delhi).*

Claim: Any subgroup of index 3 must be normal, that is, if $K < G$ such that $\left|\frac{G}{K}\right| = 3$, then $K \lhd G$.

Generalised Cayley's Theorem: If G has a subgroup of index n (say H), then there is a homomorphism f from G to S_n. Then Kernal of this homomorphism is contained in H. Moreover, if N is a normal subgroup of G such that $N \subset H$ then $N \subset Kerf$.

If G has a subgroup of index 3 then $\left|\frac{G}{H}\right| = 3$. From the Generalised Cayley's Theorem, $\exists f : G \longrightarrow S_3$. So by the First Homomorphism Theorem, $\frac{G}{Kerf} \cong f(G) < S_3$. Let $Kerf = K \implies \frac{G}{K} \cong f(G) < S_3$. Hence possibilities for $f(G)$ are $\{e\}, \{e,(12)\}, \{e,(13)\}, \{e,(23)\}, \{e,(123),(132)\}$ and S_3.

Case 1 $f(G) = S_3 \implies \frac{G}{K} \cong S_3$. Then $\frac{G}{K}$ has a subgroup $\frac{S}{K}$ whose order is 3. $o\left(\frac{G}{S}\right) = O\left(\frac{\left(\frac{G}{K}\right)}{\left(\frac{S}{K}\right)}\right) = \frac{o\left(\frac{G}{K}\right)}{o\left(\frac{S}{K}\right)} = \frac{6}{3} = 2$. (Using Third Isomorphism Theorem) This proves that G has a subgroup of index 2, which is a contradiction. Thus $f(G) \neq S_3$.

Case 2 $f(G) = \{e\} \implies \frac{G}{K} \cong \{e\} \implies G = K$. But from Generalised Cayley's Theorem $K \subset H \neq G \implies G \neq K$. Therefore $f(G) \neq \{e\}$.

Case 3 $f(G) = \{e,(12)\}/\{e,(13)\}/\{e,(23)\} \implies \left|\frac{G}{K}\right| = 2$, which is a contradiction. Therefore $f(G) \neq \{e,(12)\}/\{e,(13)\}/\{e,(23)\}$

Case 4 The only possibility is $\frac{G}{K} \cong \{e,(123),(132)\} \implies \left|\frac{G}{K}\right| = 3$ And $k = Kerf \lhd G$. Hence, any subgroup of index 3 must be normal.

Solution 9. *Solved by Amit Kumar Basistha and the proposer. The solution below is by B. Sury (Indian Statistical Institute, Bengaluru).*

One has the polynomial identity

$$\sum_{r=0}^{[(n-1)/2]} (-1)^r \binom{n-1-r}{r} x^r (1+x)^{n-1-2r} = 1 + x + \cdots + x^{n-1}$$

by induction on n. The right hand side equals $\frac{x^n-1}{x-1} = \prod_{r=1}^{n-1}(x - e^{2ir\pi/n})$. It is crying out that we combine the terms corresponding to r and $n-r$; if n is even, there is a middle term corresponding to $r = n/2$ which is $x+1$. We obtain

$$\sum_{r=0}^{(n-2)/2} \binom{n-1-r}{r} (\frac{-x}{(1+x)^2})^r (1+x)^{n-1} = (x+1) \prod_{r=1}^{\frac{n}{2}-1} (x^2 - 2xCos(2\pi r/n) + 1).$$

Let us take for x a solution of the quadratic equation $(x+1)^2 = -x$ (that is, $x^2 + 3x + 1 = 0$). Thus, one has for even n,

$$(1+x)^{n-1} \sum_{r=0}^{(n-2)/2} \binom{n-1-r}{r} = (-x)^{(n-2)/2}(1+x) \prod_{r=1}^{(n-2)/2} (3 + 2Cos(2\pi r/n)).$$

As $(1+x)^2 = -x$, we have for even n that $(1+x)^{n-1} = (1+x)(-x)^{(n-2)/2}$ which, therefore, gives the first formula :

$$F_n = \prod_{r=1}^{[(n-1)/2]} (3 + 2Cos(2\pi r/n)) \ \ \forall \ \ n \geq 1$$

where, as usual, the usual convention is that an empty product equals 1. Using this expression, here is a proof of the well-known fact F_m divides F_{mn} for all $m, n \geq 1$. In the expression

$$F_{mn} = \prod_{r=1}^{[(mn-1)/2]} (3 + 2Cos(2\pi r/mn)),$$

there are terms corresponding to $r = n, 2n, \cdots, n[\frac{m-1}{2}]$ since $n[\frac{m-1}{2}] \leq [\frac{mn-1}{2}]$. Each of these terms is also a term for F_m and, in fact, comprise of *all* the terms of F_m ! Hence F_{mn}/F_m is a product of expressions of the form $3 + 2Cos(2\pi r/mn)$. Each of these is an algebraic integer and thus, the ratio F_{mn}/F_m is simultaneously an algebraic integer and a rational number. Hence the ratio is an integer.

The solution by Amit Kumar Basistha was also along similar lines.