My First Research Article

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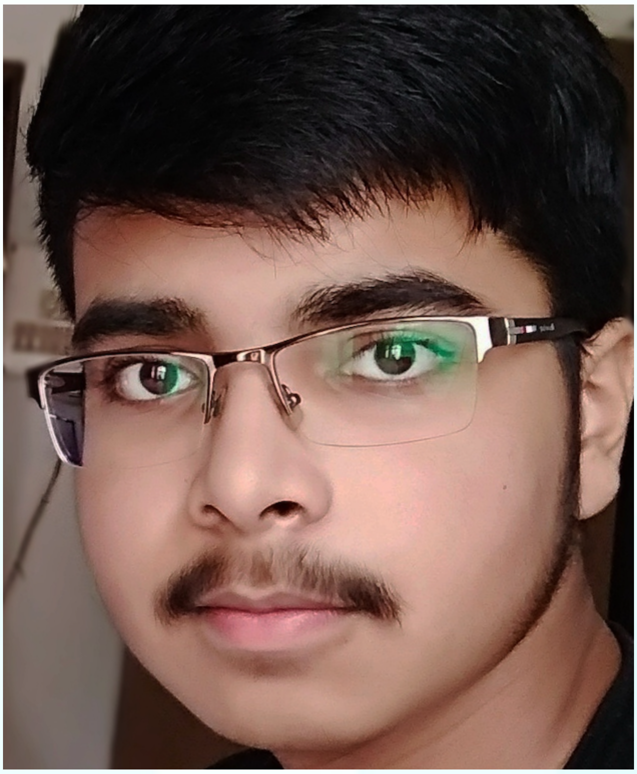
My first research article has been accepted for publication in the journal American Mathematical Monthly published by the Mathematical Association of America. Let me tell you the story behind it.



Having already qualified for IMOTC[1] 2019 in class 10, I primarily prepared for the IMO (International Mathematical Olympiad) in class 11 and became proficient in solving harder IMO problems. I was pretty good at number theory and learned analytic number theory through experience, solving problems and looking at solutions. So far I haven't read any proper analytic number theory book.

I directly qualified for IMOTC 2020 through postals. The students of IMOTC 2020 (which was held online due to the Covid-19 pandemic) decided to organise an IMO styled contest between the juniors and seniors, the juniors being those who attended IMOTC for the first time whereas I was a senior. The juniors were supposed to make a contest for the seniors and vice-versa. All the problems were to be original, forbidding copying of ideas from other possible sources. I crafted the following problem with hints of analytic number theory and proposed it for the contest.

[1] International Mathematical Olympiad Training Camp

Problem. Let $\varepsilon > 0$ be a constant. If a and b are sufficiently large positive integers such that $a! + b! \mid (a+b)!$ then prove that $a < (1+\varepsilon)b$.

It received many positive comments from people whom I considered my idols. However it got rejected, the reason being that it was "too hard" for the contest. I was enervated. Ignited by the rejection of the problem, I made it a mission to make analytic number theory popular among olympiad contestants. With days of work I prepared an olympiad handout on the topic of analytic number theory, and posted it online on AoPS (Art of Problem Solving). Within hours it received a lot of positive feedback. But my proposed problem and days of hard work going to vain was tough to digest.

I decided to perform a complete analysis of the divisiblity $a! + b! \mid (a+b)!$. So let us begin; I will try my best to explain the thought process along with the motivations behind the results.

Call a pair (a, b) of positive integers satisfying the divisiblity condition *good*. A graph of all the good pairs (a, b) where $1 \leq a \leq b \leq 100$ tells a lot about them.

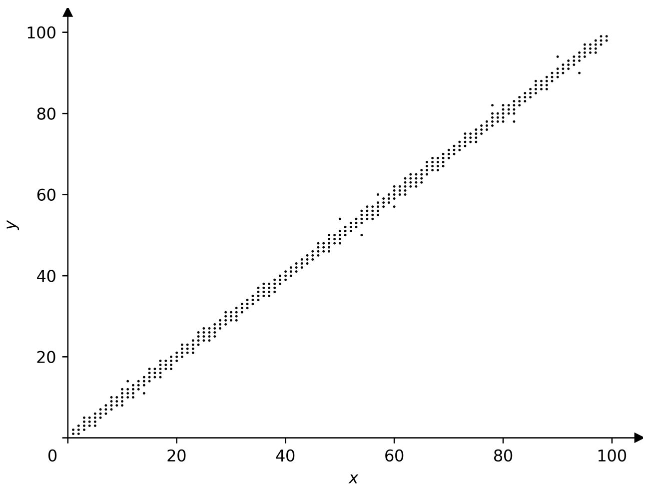


Figure 1. Plot of good pairs.

We immediately observe that the graph resembles the line $x = y$. This motivates us to study the quantity $a - b$ for good pairs (a, b); it can be thought of as a measure of how far a point (a, b) is from the line $x = y$. The following result shows a fairly good upper bound on the quantity $a - b$.

Theorem 1. *If a and b are positive integers such that $a \geq b \geq e^{e^{e^{4.22}}}$ and $a! + b!$ divides $(a+b)!$, then*

$$a - b < \frac{b \log \log b}{\log b}.$$

The constant $c = e^{e^{e^{4.22}}}$ may look intimidating at first sight, but it is just a result of some ugly calculations, not really something to care about, the existence of such a constant c being the main point. Using the above theorem on sufficiently large a and b with $a \geq b$, we have that

$b \le a \le b + \dfrac{b \log\log b}{\log b}$. Dividing by b, we obtain

$$1 \le \frac{a}{b} \le 1 + \frac{\log\log b}{\log b}.$$

From here it becomes clear that the ratio a/b converges to 1 since $\log b$ grows faster than $\log\log b$. This explains why the graph resembles the $x = y$ line (see Figure 1).

I also found out all good pairs of Fibonacci numbers since this is what mathematicians usually do when they can't solve a problem in it's generality - consider a special case. Although it should be obvious that the divisiblity cannot be solved in it's full general form.

Theorem 2. *If a and b are Fibonacci numbers such that $a! + b! \mid (a+b)!$, then*

$$(a, b) \in \{(2,1), (3,2), (5,3), (F_n, F_n)\}$$

up to permutation. Here F_n denotes the nth Fibonacci number.

Once we prove the upper bound on the quantity $|a - b|$ (See Theorem 1), a natural question is whether the quantity can get arbitrarily large. Or even better, can the quantity be any positive integer? It is trivial that (n, n) and $(n, n+1)$ are good pairs for all positive integers n. The following theorem implies that there are infinitely many good pairs of the form $(n, n+2)$.

Theorem 3. *The pair $s = (x^2(x^2+2), x^2(x^2+2)+2)$ is good for all positive integers x.*

The choice of the pair s may seem cryptic, however the motivation is very simple. For $(n, n+2)$ to be a good pair, we require $n! + (n+2)! \mid (2n+2)!$, which is equivalent to $(n+1)(n+2)+1 \mid \frac{(2n+2)!}{n!}$. Now we want to choose n so that the polynomial $(n+1)(n+2)+1$ factors nicely, as a consequence of which the LHS has small prime factors and the chances of the LHS dividing the RHS is high. The substitution $n = x^2(x^2+2)$ is our saviour! I believe that the following is also true.

Conjecture. For any positive integer k, there are infinitely many good pairs of the form $(n, n+k)$.

The conjecture is proved for $k = 2, 3, 4$.

I put together these results along with their proofs into an article and submitted it to the American Mathematical Monthly as I had nothing to lose. After a long wait I got an email on 24th December, 2020 from the editor of the American Mathematical Monthly that my paper[2] titled "On the Divisibility $a! + b! \mid (a+b)!$" was accepted for publication!

[2] `https://www.dropbox.com/s/1e96zp2m1eaukmn/amm-paper.pdf?dl=1`