5 Problems 1 Solution : $a^n + b^n$ is a multiple of $a + b$ for $n = 1, 3, 5, \ldots$

Pankaj Agarwal

E-mail: `pamathsfac@gmail.com`

Under the title '5 Problems 1 Solution', we intend to discuss, 5 problems which can be solved using the same concept. We intend to keep the concept basic. In this particular article, the concept we will be using is as follows:

If a and b are integers such that $a + b \neq 0$ and n is an odd positive integer, then $a^n + b^n$ is always a multiple of $a + b$.

The above result can be proved in various ways. Some of them are

(i) using geometrical progression concept,

(ii) using polynomial (Factor Theorem),

(iii) using multiplication theorem.

The intersecting part is that how such a simple concept can be used to solve various type of Math Olympiad Problems.

Problem 1. Prove that $4^n + 2$ is divisible by 6 for every positive integer n.

(The Talent Search – South Africa, 1994)

Solution: We have,

$$\begin{aligned} 4^n + 2 &= 2^{2n} + 2 \\ &= 2(2^{2n-1} + 1) \\ &= 2 \times \text{a multiple of } (2+1), \ \text{[since } 2n-1 \text{ is odd]} \\ &= \text{a multiple of } 6. \end{aligned}$$

Problem 2. Prove that if p and q are primes and $q = p + 2$, then $p^q + q^p$ is divisible by $p + q$.

(Moscow, 1968)

Solution: Since, $q = p + 2$ and p, q are primes, so q and p both are odd. Now,

$$\begin{aligned} p^q + q^p &= p^{p+2} + q^p \\ &= (p^2 - 1)p^p + (p^p + q^p) \\ &= (p - 1)(p + 1)p^p + (p^p + q^p). \end{aligned}$$

Now, $p^p + q^p$ is a multiple of $p + q$ as p is odd. We have $2(p + 1) = p + (p + 2) = p + q$. So, as $p - 1$ is even, we get $(p - 1)(p + 1)$ is divisible by $p + q$.

So, the given expression is a multiple of $p + q$.

Problem 3. Let p be an odd positive integer. Show that $1^p + 2^p + 3^p + \cdots + 100^p$ is divisible by $1 + 2 + 3 + \cdots + 100$.

Solution: Let $f(p) = 1^p + 2^p + 3^p + \cdots + 100^p$.

Now, $1 + 2 + 3 + \cdots + 100 = 50 \times 101$. So we need to show that $f(p)$ is divisible by both 50 and 101.

Firstly, we observe that each of $1^p + 100^p, 2^p + 99^p, \ldots, 50^p + 51^p$ is divisible by 101. So, $f(p)$ is divisible by 101.

Secondly, $f(p) = 50^p + 100^p + (1^p + 99^p) + (2^p + 98^p) + \cdots + (49^p + 51^p)$. We observe that each of $(1^p + 99^p), (2^p + 98^p), \ldots, (49^p + 51^p)$ is divisible by 100. So, $f(p)$ is also divisible by 50.

Problem 4. Prove that for every positive integer n, the number $a_n = 5^n + 2 \times 3^{n-1} + 1$ is a multiple of 8.

(Eotvos Competition, 1912)

Solution: We consider the two cases: (1) n is odd, and (2) n is even.

Case 1: Let $n = 2k + 1$. Then,

$$\begin{aligned} a_n &= 5^n + 2 \times 3^{n-1} + 1 \\ &= 5^n + (3 - 1) \times 3^{n-1} + 1 \\ &= (5^n + 3^n) - (3^{n-1} - 1) \\ &= (5^n + 3^n) - (3^k - 1)(3^k + 1). \end{aligned}$$

Now, $5^n + 3^n$ is a multiple of $5 + 3$ as n is odd. Also, $3^k – 1$ and $3^k + 1$ are two consecutive even numbers, and so one of them must be a multiple of 4. So, their product is a multiple of 8.

Case 2: Let $n = 2k$. Then,

$$\begin{aligned}a_n &= 5^n + 2 \times 3^{n-1} + 1\\ &= 5^n + (5-3) \times 3^{n-1} + 1\\ &= 5(5^{n-1} + 3^{n-1}) - (3^n - 1)\\ &= 5(5^{n-1} + 3^{n-1}) - (3^k - 1)(3^k + 1).\end{aligned}$$

Now, as $n - 1$ is odd, we get $5^{n-1} + 3^{n-1}$ is divisible by $5 + 3$.

Problem 5. What is the largest positive integer n for which $n^3 + 100$ is divisible by $n + 10$?

(AIME, 1986)

Solution: We have, $n^3 + 100$ is divisible by $n + 10$. Also $n^3 + 1000$ is divisible by $n + 10$. So, $(n^3 + 1000) - (n^3 + 100) = 900$ is divisible by $n + 10$.

Let $k(n + 10) = 900$. Then, for n to be greatest, k must be the least possible positive integer. So, putting $k = 1$, we have $n = 890$ is the greatest possible integral value.

Exercise:

Besides these 5 problems, there are many problems that can be solved using the same concept as above. A few problems in this category are given below.

1) Given x, y, z are primes such that $x^y + 1 = z$. Find x, y, z. *(Assam Maths Olympiad, 2015)*
[Hints: If y is odd, then $z = x^y + 1^y$ is a multiple of $x + 1$, and so z cannot be prime. We get, y is even, i.e. $y = 2$.]

2) Show that $3333^{4444} + 4444^{3333}$ is divisible by 7777.
[Hints: Both 3332 and 4445 are divisible by 7.]

3) Given unequal integers x, y, z. Prove that $(x-y)^5 + (y-z)^5 + (z-x)^5$ is divisible by $5(x-y)(y-z)(z-x)$. *(All soviet Union, 1967)*
[Hints: (a) $(x-y)^5 + (y-z)^5$ is multiple of $(x-y) + (y-z) = z - x$.
(b) To prove that the given expression is divisible by 5, use Fermat's Little Theorem, since the expression is equal to $[(x-y)^5 - (x-y)] + [(y-z)^5 - (y-z)] + [(z-x)^5 - (z-x)]$.]

4) Determine all positive integers n for which $2^n + 1$ is divisible by 3. *(Eotvos Competition, 1898)*
[Hints: Consider the two cases: (1) n is odd, and (2) n is even.]

5) Prove that, for any positive integer n, $1^n + 2^n + 3^n + 4^n$ is divisible by 5 if and only if n is not divisible by 4. *(Eotvos Competition, 1901)*
[Hints: Consider the two cases: (1) n is odd, and (2) n is even. Then consider the two forms $n = 4k + 2$ and $n = 4k$.]