5 Problems 1 Solution : Square of an integer cannot be of $3k+2$ type

Pankaj Agarwal

E-mail: pamathsfac@gmail.com

Under the title '5 Problems 1 Solution', I intend to discuss 5 problems, which can be solved using the same concept. I intend to keep the concept basic. In this particular article, the concept we will be using is as follows:

If n is any integer, then either n or n^2-1 is a multiple of 3. This can also be considered as "the square of any integer cannot be of $3k+2$ type for any integer k". The interesting part is that how such a simple concept can be used to solve various type of Mathematics Olympiad problems.

Problem 1. Find all prime numbers p such that $p^2+2007p-1$ is prime as well.

(Berkeley Math circle - 2008)

Solution: If $p \neq 3$, then p is not a multiple of 3 (since p is prime). Then p^2-1 is a multiple of 3. So, $(p^2-1)+2007p$ is a multiple of 3 (as $2007 = 3 \times 3 \times 223$) and hence $p^2+2007p-1$ cannot be prime. Also, if $p=3$, then $p^2+2007p-1 = 9+6021-1 = 6029$, which is a prime.

Problem 2. How many primes p are there such that $2p^4-7p^2+1$ is equal to the square of an integer?

(Turkey National Olympiad, 2001 - Round 1)

Solution: If $p \neq 3$, then p is not a multiple of 3 (as p is prime). Then p^2-1 and p^4-1 are both multiples of 3. So,

$$\begin{aligned} 2p^4-7p^2+1 &= 2(p^4-1)-7(p^2-1)+1+2-7 \\ &= [2(p^4-1)-7(p^2-1)-6]+2 \\ &= (a\ multiple\ of\ 3)+2. \end{aligned}$$

But the square of any integer cannot be of $3k + 2$ type. So, $p \neq 3$ is impossible. Hence, $p = 3$ is the only possibility.

Problem 3. If p and $p^2 + 2$ are prime numbers, then find the number of prime divisor of $p^3 + 3$.

(Turkey National Olympiad, 2006 - Round 1)

Solution: If $p \neq 3$, then p is not a multiple of 3 (since p is prime) and hence $p^2 - 1$ is a multiple of 3. So, $p^2 + 2 = (p^2 - 1) + 3$ is a multiple of 3. But $p^2 + 2$ is prime. So, $p \neq 3$ is not possible. So, $p = 3$ is the only possibility, and hence $p^3 + 3 = 30 = 2 \times 3 \times 5$, which has 3 prime divisors 2, 3 and 5.

Problem 4. Show that the equation $x^2 - 3y^2 = 17$ has no integral solutions.

(Berkeley Math circle)

Solution: As 17 is not a multiple of 3, so x is not a multiple of 3. Therefore, $x^2 - 1$ is a multiple of 3. Hence, the equation $(x^2 - 1) - 3y^2 = 16$ is not possible as LHS is a multiple of 3 and RHS is not a multiple of 3.

Problem 5. Can 2004^{2005} be written as the sum of two perfect squares?

(Italy, 2004)

Solution: If possible, suppose that $2004^{2005} = a_0^2 + b_0^2$, where a_0 and b_0 are integers.

But, 2004^{2005} is a multiple of 3. So, a_0 and b_0 must both be multiples of 3. Put $a_0 = 3a_1$, and $b_0 = 3b_1$. Then we get, $2004^{2003} \cdot 668 = a_1^2 + b_1^2$.

Using the same logic, we see that both a_1 and b_1 are multiples of 3. So, taking $a_1 = 3a_2$ and $b_1 = 3b_2$, we get $2004^{2001} \cdot 668^2 = a_2^2 + b_2^2$.

Continuing in this manner and taking $a_k = 3a_{k+1}$ and $b_k = b_{k+1}$, we get $2004 \cdot 668^{1002} = a_{1002}^2 + b_{1002}^2$. So, both a_{1002} and b_{1002} must be multiples of 3.

So, $668^{1003} = 3(a_{1003}^2 + b_{1003}^2)$. Which is not possible as LHS is not a multiple of 3 while RHS is a multiple of 3.

Besides these 5 problems, there are many problems that can be solved using the same concept as above. Of course, in some of them some other concepts may also be required. The questions given below are in this category.

1) Find all positive prime numbers p, q, r, s such that $p^2 + 2019 = 26(q^2 + r^2 + s^2)$.

(Conosur, 2019)

[Hint : If $p = 3$, then $q^2 + r^2 + s^2 = 78$, which can be easily solved. If $p \neq 3$, then $q^2 + r^2 + s^2$ is $3k + 2$ type. So, one of q, r, s must be multiples of 3. But p, r, s are primes. So, one of them must be 3.]

2) Find all prime numbers p such that $2^p + p^2$ is also a prime number.

(Albania - TST, 2011)

[Hint: Obviously p is odd. Use the concept that $a^n + b^n$ is divisible by $a + b$ whenever n is odd. Observe that $2^p + p^2 = (2^p + 1^p) + (p^2 - 1)$.]

3) Find all primes p such that the number $p^2 + 11$ has exactly six divisors (including 1 and the number itself).

(Russia - 1995)

[Hint: If $p \neq 3$, then $p^2 + 11$ is a multiple of 3. Also, $p = 2$ does not satisfy the conditions. Now, show that $p^2 + 11$ is also a multiple of 4.]

4) Find all prime numbers a, b, c and positive integer k, which satisfy the equation $a^2 + b^2 + 16c^2 = 9k^2 + 1$.

(JBMO - 2015)

[Hint: Observe that $a^2 + b^2 + c^2$ is of $3k + 1$ type. So, exactly two of a, b, c are multiples of 3. But a, b, c being prime, we know exactly 2 of them equals 3.]

5) Find all prime numbers p such that $5^p + 12^p$ is a perfect square.

(Singapore TST - 1999/2000)

[Hint: It p is odd, then $5^p + 12^p = [(5^p + 1^p) + 12^p - 3] + 2$ is of $3k + 2$ type, which is not possible. So, p is even.]