A Taste of Analytic Number Theory, Part III[1]

Ayan Nath

HS Student, Kaliabor College

Abstract. These series of articles (three in total) are aimed at olympiad contestants, focuses on solving olympiad Number Theory problems using analytic techniques and making contestants familiar with common techniques and results in this topic. We started with the Prime Number Theorem, giving an elementary proof of the weak version and establishing a few well known estimates for the two Chebyshev functions. We also showed Mertens' first theorem on the fly and discussed Mertens' second theorem, asymptotic density and equidistribution theorem. In this concluding part we will present some problems.

7. Problems

All the problems below don't necessarily use the theory discussed in this series of articles. Many of the following problems are hard so don't get demotivated.

7.1. Exercises

If you are experienced then you may skip this section.

Exercise 7.1. Let A be a set of positive integers with positive asymptotic density. Prove that sum of reciprocals of elements of A is divergent.

Exercise 7.2. If the density of $A \subset \mathbb{N}$ and $B \subset \mathbb{N}$ is zero then prove that density of $A \cup B$ is zero.

Exercise 7.3. You are given a string of base-10 digits. Prove that you can append some finite number of digits so that the resultant number becomes a power of 2.

Exercise 7.4. Define $\omega(n)$ to be the number of distinct prime divisors of n. Prove that

$$\sum_{n \leq x} \omega(n) = x \log\log x + \mathcal{O}(x).$$

[1] **Editor's Note:** Part I (in Volume 67) contained section 1 and Part II (in Volume 68) contained sections 2 through 6.

Exercise 7.5. Prove that

$$\prod_p \left(1 - \frac{1}{p}\right) = 0,$$

where the product is over all primes p.

Exercise 7.6. Let $r_2(n)$ be the number of ways n can be written as a sum of two perfect squares. Prove that

$$\lim_{n\to\infty} \frac{r_2(1) + r_2(2) + \cdots + r_2(n)}{n} = \pi.$$

Exercise 7.7 (Mathotsav)**.** We say that a positive integer t is good if the density of positive integers n such that $n^2 + t$ is square-free is at least 0.99.

(a) Prove that the density of square free numbers is $\frac{6}{\pi^2}$.

(b) Prove that infinitely many natural numbers are good.

(c) Prove that there exists a positive constant c and a natural number N, such that for all $n > N$, the number of natural numbers less than n which are good is at least cn.

7.2. Easy

Problem 7.1 (Iranian Our MO 2020)**.** Consider two sequences $x_n = an + b$, $y_n = cn + d$ where a, b, c, d are natural numbers and $\gcd(a, b) = \gcd(c, d) = 1$, prove that there exist infinite n such that x_n, y_n are both square-free.

Problem 7.2 (Iran 3rd round 2010/8)**.** Prove that there are infinitely many natural numbers of the form $n^2 + 1$ such that they don't have any divisor of the form $k^2 + 1$ except 1 and themselves.

Problem 7.3 (China TST 2005)**.** Prove that for any n ($n \geq 2$) pairwise distinct fractions in the interval $(0, 1)$, the sum of their denominators is no less than $\dfrac{1}{3}n^{\frac{3}{2}}$.

Problem 7.4 (China TST 2004)**.** Let u be a fixed positive integer. Prove that the equation $n! = u^\alpha - u^\beta$ has a finite number of solutions (n, α, β).

Problem 7.5 (IMO Shortlist 2011/A2)**.** Determine all sequences $(x_1, x_2, \ldots, x_{2011})$ of positive integers, such that for every positive integer n there exists an integer a with

$$\sum_{j=1}^{2011} j x_j^n = a^{n+1} + 1.$$

Problem 7.6 (China TST 2010, Miklos Schweitzer, Paul Erdos)**.** Given positive integers n and k such that $n \geq 9^k$, prove that $\binom{n}{k}$ has at least k different prime divisors.

Problem 7.7 (IMO ShortList 2003/N4)**.** Let b be an integer greater than 5. For each positive integer n, consider the number

$$x_n = \underbrace{11\cdots1}_{n-1}\underbrace{22\cdots2}_{n}5,$$

written in base b.

Prove that the following condition holds if and only if $b = 10$: there exists a positive integer M such that for any integer n greater than M, the number x_n is a perfect square.

Problem 7.8 (Vesselin Dmitrov)**.** Prove that the set of positive integers n such that

$$\tfrac{1}{2}n(n+1)(n+2)(n^2+1)$$

is square free has positive density.

Problem 7.9 (Miklos Schweitzer)**.** Prove that the set of positive integers n such that $\tau(n) \mid n$ has density 0.

7.3. Medium

Problem 7.10 (ARMO 2012 Grade 11 Day 2)**.** For a positive integer n define

$$S_n = 1! + 2! + \ldots + n!.$$

Prove that there exists an integer n such that S_n has a prime divisor greater than 10^{2012}.

Problem 7.11 (AoPS)**.** Prove that $n! = m^3+8$ has only finitely many solutions in positive integers.

Problem 7.12 (China TST 2 Day 1 P1)**.** Let n be a positive integer. Let D_n be the set of all divisors of n and let $f(n)$ denote the smallest natural m such that the elements of D_n are pairwise distinct in mod m. Show that there exists a natural N such that for all $n \geq N$, one has $f(n) \leq n^{0.01}$.

Problem 7.13 (Paul Erdos, Miklos Schweitzer)**.** Let $a_1 < a_2 < \cdots < a_n$ be a sequence of positive integers such that $a_i - a_j \mid a_i$ for all $i \leq j$. Prove that there is a positive constant c such that for any such sequence of length n, $a_1 > n^{cn}$.

Problem 7.14. (Tuymaada 2011, Senior Level) Let $P(n)$ be a quadratic trinomial with integer coefficients. For each positive integer n, the number $P(n)$ has a proper divisor d_n, i.e., $1 < d_n < P(n)$, such that the sequence $d_1, d_2, d_3, \ldots$ is increasing. Prove that either $P(n)$ is the product of two linear polynomials with integer coefficients or all the values of $P(n)$, for positive integers n, are divisible by the same integer $m > 1$.

Problem 7.15. (Turkey TST 2015/6) Prove that there are infinitely many positive integers n such that $(n!)^{n+2015}$ divides $(n^2)!$.

Problem 7.16 (China TST 2015)**.** Let $a_1, a_2, a_3, \ldots$ be distinct positive integers, and $0 < c < \frac{3}{2}$. Prove that: There exist infinitely many positive integers k, such that $\operatorname{lcm}(a_k, a_{k+1}) > ck$.

Remark 7.1. The bound cannot be improved to $\text{lcm}(a_k, a_{k+1}) > k^{1+\delta}$ for some $\delta > 0$.

Problem 7.17 (USA TSTST 2017/6)**.** A sequence of positive integers $(a_n)_{n\geq 1}$ is of Fibonacci type if it satisfies the recursive relation $a_{n+2} = a_{n+1} + a_n$ for all $n \geq 1$. Is it possible to partition the set of positive integers into an infinite number of Fibonacci type sequences?

Problem 7.18 (Tuymaada 2007/8)**.** Prove that there exists a positive c such that for every positive integer N among any N positive integers not exceeding $2N$ there are two numbers whose greatest common divisor is greater than cN. (Bonus: Strengthen the bound.)

7.4. Hard

Problem 7.19 (IMO 2015/N6)**.** Let $\mathbb{Z}_{>0}$ denote the set of positive integers. Consider a function $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$. For any $m, n \in \mathbb{Z}_{>0}$ we write $f^n(m) = \underbrace{f(f(\dots f}_{n}(m)\dots))$. Suppose that f has the following two properties:

(i) if $m, n \in \mathbb{Z}_{>0}$, then $\frac{f^n(m)-m}{n} \in \mathbb{Z}_{>0}$;
(ii) The set $\mathbb{Z}_{>0} \setminus \{f(n) \mid n \in \mathbb{Z}_{>0}\}$ is finite.

Prove that the sequence $f(1) - 1, f(2) - 2, f(3) - 3, \dots$ is periodic.

Problem 7.20 (China TST 2018 Day 2 Q2)**.** Given a positive integer k, call n good if among

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \dots, \binom{n}{n}$$

at least $0.99n$ of them are divisible by k. Show that exists some positive integer N such that among $1, 2, \dots, N$, there are at least $0.99N$ good numbers.

Problem 7.21 (Paul Erdos)**.** For any $\delta > 0$ prove that there are at least $(\frac{2}{3} - \delta)\dfrac{n}{\log_2 n}$ primes between n and $2n$ for sufficiently large n. (Using the full power of PNT would be cheating.)

Problem 7.22 (XIII Brazilian Olympic Revenge 2014)**.** Let $a > 1$ be a positive integer and $f \in \mathbb{Z}[x]$ with positive leading coefficient. Let S be the set of integers n such that

$$n \mid a^{f(n)} - 1.$$

Prove that S has density 0; that is, prove that $\lim_{n\to\infty} \dfrac{|S \cap \{1, \dots, n\}|}{n} = 0$.

Problem 7.23 (PRIMES 2020 M5)**.** We say an integer $n \geq 2$ is chaotic if for any monic nonconstant polynomial $f(x)$ with positive integer coefficients, the set

$$\{f(1), f(2), \dots, f(n)\}$$

contains fewer than $10^{\deg f} \cdot \dfrac{n}{\log n}$ prime numbers. Are there finitely many chaotic integers?

Remark 7.2. There is a theorem by Nagell & Heilbronn which says that for any $f \in \mathbb{Z}[x]$, the number of primes in $\{|f(1)|, |f(2)|, \ldots, |f(n)|\}$ is $\mathcal{O}(n/\log n)$ but unfortunately the proof is beyond the scope of Olympiad Mathematics.

Problem 7.24 (Marius Cavachi, AMM)**.** Let a and b be integers greater than 1 such that $a^n - 1 \mid b^n - 1$ for every positive integer n. Prove that b is a natural power of a.

Remark 7.3. You can relax the condition to "for infinitely many positive integers n" instead of "for every positive integer n" and the problem would still hold. However the proof of this is non-elementary.

Problem 7.25 (Fedor Petrov)**.** Does there exist $c > 0$ such that among any n positive integers one may find 3 with least common multiple at least cn^3?

8. Solutions to selected examples

8.1. Example 1.12

Let's suppose we want $g(n) = k$. Choose a large enough natural t such that $2^t < q < 2^t(1+\varepsilon)$ where q is a prime. Note that $n = q^{2k}2^{2kt}$ works because all such k divisors are of the form $q^{k+i}2^{t(k-i)}$ for $i = 1, 2, \ldots, k$. No other divisor works because for any fixed power of q we can have only one power of 2 which may work.

8.2. Example 1.15 (EMMO 2016 Sr, Anant Mudgal)

Part (b) is easy so we only solve part (a). Assume the contrary that all sufficiently large indices are divisor friendly. We have that

$$a_n \nmid \operatorname{lcm}(a_1, a_2, \ldots, a_{n-1})$$

for all $n > K$, say. Observe that there must exist some sequence of primes q_n such that if $b_n = q_n^{\nu_{q_n}(a_n)}$ for $n > K$ then q_n divides none of the preceding terms a_i for $i > K$. See that all the b_n's must be distinct. Obviously $b_1, b_2, \ldots, b_n \leq 9000n$ and b_i are distinct prime powers. Number of prime powers at most $9000n$ is less than

$$\begin{aligned}
S = \sum_{p \leq 9000n} \log_p 9000n &= \log 9000n \sum_{p \leq 9000n} \frac{1}{\log p} \\
&\leq \log 9000n \left(\sum_{p < \sqrt{9000n}} \frac{1}{\log p} + \sum_{\sqrt{9000n} \leq p \leq 9000n} \frac{1}{\log p} \right) \\
&\leq \log 9000n \left(\frac{\sqrt{9000n}}{\log 2} + \frac{1}{\log \sqrt{9000n}} \cdot \left(\pi(9000n) - \pi(\sqrt{9000n})\right) \right) \\
&= \log 9000n \left(\frac{\sqrt{9000n}}{\log 2} + \frac{1}{\log \sqrt{9000n}} \cdot \mathcal{O}\left(\frac{n}{\log n}\right) \right) \\
&= \log 9000n \cdot \mathcal{O}\left(\frac{n}{\log^2 n}\right) = \mathcal{O}\left(\frac{n}{\log n}\right),
\end{aligned}$$

in the last second step we used PNT. This is a contradiction for large enough n since there are n distinct prime powers at most $9000n$, namely $b_1, b_2, \dots, b_n$. And we are done.

8.3. Example 5.3 (Canada MO 2020/4)

Consider $(9999n+4999)^2$ and $(9999n+5000)^2$, verify that their difference is divisible by 9999, call a pair of such perfect squares good. Fix some large N. Check that the number of such pairs less than N is bounded below by $c\sqrt{N}$ for some constant $c > 0$. All perfect powers between such a pair must be odd perfect powers. Number of odd perfect powers a^b less than N is at most

$$S = N^{1/3} + N^{1/5} + N^{1/7} + \cdots ,$$

where the number of summands is at most $\log_2 N$ as $a \geq 2$ except for the trivial perfect power 1. Therefore $S = \mathcal{O}(N^{1/3} \log N)$, which is less than $c\sqrt{N}$ for all large N. Thus there exists infinitely many good pairs.

8.4. Example 5.6 (Iran 3rd round 2011)

(Solution by `Superguy`) We are going to prove the bound $q_n \leq 35^n$ for part (a). Let's assume for the sake of contradiction that there exists n such that $q_n > 35^n$, here suppose n is minimal.Then suppose r is the minimal index such that $q_n \mid a_r$ then $r > 35^{\frac{2}{3}n}$(♣). So all of $\{a_1, a_2, \dots, a_{r-1}\}$ have prime factors in set $\{q_1, q_2, \dots, q_{n-1}\}$. Call this set of primes as P. We clearly have

$$\sum_{k=1}^{r-1} \frac{1}{a_k^{\frac{1}{3}}} \geq \sum_{k=1}^{r-1} \frac{1}{\sqrt{k}}. \tag{1}$$

Clearly RHS in (1) is greater than $2\sqrt{r}-2$ which can be shown using easy integration or induction. Consider the following claim.

Claim.

$$\sum_{k=1}^{r-1} \frac{1}{a_k^{\frac{1}{3}}} \leq \prod_{p\in P}\Big[\sum_{m\geq 0} p^{-\frac{1}{3}m}\Big] \leq 5(3.27)^{n-2},$$

where $|P| = n-1$.

Proof. Note that all of a_k are of the form $q_1^{k_1} \cdot q_2^{k_2} \cdots q_{n-1}^{k_{n-1}}$ where all k_i are non-negative which gives the left side inequality. For right side we have that the sum$\sum_{m\geq 0} p^{-\frac{1}{3}m}$ is maximum for $p=2$ and next greatest value is achieved by $p=3$ and the value of the sum for $p=2$ is less than 5 and for $p=3$ the sum would be less than 3.27 Now observe

$$\prod_{p\in P}\Big[\sum_{m\geq 0} p^{-\frac{1}{3}m}\Big] \leq 5(3.27)^{n-2}.$$

So we get the claim. □

Now by our claim, (1),(♣) and the fact that

$$\ln(2\cdot 35^{\frac{n}{3}}) < \ln(2\cdot 35^{\frac{n}{3}} - 2) + 1 \text{ for all natural } n,$$

we get that we should have

$$\ln(2) - 1 + \frac{n \cdot \ln(35)}{3} - \ln(5) - (n-2)(\ln(3.27)) < 0. \tag{2}$$

Now we are going to prove the opposite inequality. Taking the function in LHS as $f(n)$ we get that $f(n)$ is increasing. Hence we just need to check for $n = 1$ which we get that $f(1) > 0$. Thus we have proved the opposite inequality and thus the contradiction for our initial assumption. For part (b) exact similar process can give a nice bound of some $q_n < 300^n$. ■

8.5. Example 5.7 (IMO 2008/3 improved)

Define

$$f(N) = \prod_{n \le N} (n^2 + 1).$$

Let us assume that the largest prime divisor of $f(N)$ is t. Let $f(N) = \prod_p p^{\alpha_p}$ be the prime factorisation of $f(N)$, each prime $p > N$ can divide $n^2 + 1$ for at most two different values of n, and so $\alpha_p \le 2$ in this case. See that $\alpha_2 = \lfloor N/2 \rfloor$. For $p \le x$, if $p \mid n^2 + 1$, then $n^2 \equiv -1 \bmod p$ which has solutions if and only if $p \equiv 1 \pmod 4$, and in that case there will be at most $2\lceil N/p \rceil$ values of n for which $p \mid n^2 + 1$. Similarly, if $p^k \mid n^2 + 1$, then $n^2 \equiv -1 \bmod p^k$, and there are at most 2 solutions to this congruence and hence at most $2\lceil N/p^k \rceil$ values of n for which $p^k \mid n^2 + 1$. Combining, we find that for $p \le N$ and $p \equiv 1 \pmod 4$

$$\alpha_p \le 2\left\lceil \frac{N}{p} \right\rceil + 2\left\lceil \frac{N}{p^2} \right\rceil + 2\left\lceil \frac{N}{p^3} \right\rceil + \cdots + 2\left\lceil \frac{N}{p^k} \right\rceil,$$

where $k = \lceil \log_p N \rceil$. This gives that

$$\alpha_p \le \frac{2N}{p-1} + 2\left(\log_p N + 1\right),$$

since $1 + \frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{p^k} \le \frac{1}{1 - 1/p}$. Thus,

$$f(N) \le 2^{N/2} \prod_{\substack{p \le N \\ p \equiv 1 \pmod 4}} p^{\frac{2N}{p-1} + 2\log_p(N) + 2} \prod_{\substack{N < p \le t \\ p \equiv 1 \pmod 4}} p^2,$$

and so,

$$f(N) \le 2^{N/2} \prod_{\substack{p \le N \\ p \equiv 1 \pmod 4}} N^2 \prod_{\substack{p \le N \\ p \equiv 1 \pmod 4}} p^{\frac{2N}{p-1}} \prod_{\substack{p \le t \\ p \equiv 1 \pmod 4}} p^2.$$

Taking the logarithm

$$\log f(N) \le 2 \log N \sum_{\substack{p \le N \\ p \equiv 1 \pmod 4}} 1 + 2N \sum_{\substack{p \le N \\ p \equiv 1 \pmod 4}} \frac{\log p}{p-1} + \sum_{\substack{p \le t \\ p \equiv 1 \pmod 4}} \log p + \frac{N}{2} \log 2.$$

By PNT for AP and with some computations we can see that the RHS is asymptotic to $N \log N + t$. Notice that

$$f(N) \ge \prod_{n \le N} n^2 = (N!)^2 = N^{2N} + \mathcal{O}(N),$$

combining, we get that

$$2N\log N + \mathcal{O}(N) \le \log f(N) \le N\log N + t + o(N\log N),$$

now if $t \le (1-\varepsilon)N\log N$ for all large N then the above is false for sufficiently large N, which is what we wanted.

8.6. Example 5.10 (STEMS 2020 B3/C5, Arka Karmakar)

Clearly $b \neq 1$. Note that the leading coefficient of f must be positive and $f \in \mathbb{Q}[x]$. For now assume that f is non-constant. Consider the following claims.

Claim 1. For $n, i \in \mathbb{N}$ let $f(a^n + a^i) = b^{t(n,i)} + b^{m(n,i)}$ where $t(n,i) \ge m(n,i)$. And let $i_1, i_2, i_3, \ldots, i_k \in \mathbb{N}$, then it follows that $t(n,i_1) = t(n,i_2) = \ldots = t(n,i_k)$ for all large n.

Proof. Let $i > j$ be two positive integers. It is obvious that $t(n,i) \ge t(n,j)$ for all large n. Observe that

$$\begin{aligned}
1 = \lim_{n\to\infty} \frac{f(a^n + a^i)}{f(a^n + a^j)} &= \lim_{n\to\infty} \frac{b^{t(n,i)} + b^{m(n,i)}}{b^{t(n,j)} + b^{m(n,j)}} \\
&= \lim_{n\to\infty} \frac{1 + b^{m(n,i)-t(n,i)}}{b^{t(n,j)-t(n,i)} + b^{m(n,j)-t(n,i)}} \\
&\ge \lim_{n\to\infty} \frac{1}{b^{-(t(n,i)-t(n,j))} + b^{-(t(n,i)-m(n,j))}} \\
&\ge \lim_{n\to\infty} \frac{1}{2b^{-(t(n,i)-t(n,j))}} \\
&= \frac{1}{2} \lim_{n\to\infty} b^{t(n,i)-t(n,j)}.
\end{aligned}$$

If $b > 2$ we get $t(n,i) = t(n,j)$ for all sufficiently large n. So let $b = 2$ then either $t(n,i) = t(n,j)$ for all sufficiently large n or $t(n,i) = t(n,j) + 1$ for all sufficiently large n. We assume the later. Then note that,

$$1 = \lim_{n\to\infty} \frac{b^{t(n,i)} + b^{m(n,i)}}{b^{t(n,j)} + b^{m(n,j)}} = \lim_{n\to\infty} \frac{2 + 2^{m(n,i)-t(n,j)}}{1 + 2^{m(n,j)-t(n,j)}} \ge \lim_{n\to\infty} \frac{2}{1 + 2^{m(n,j)-t(n,j)}},$$

which implies that $m(n,j) = t(n,j)$ for all large n. Let $i > j > 1$. Hence we obtain

$$\begin{aligned}
f(a^n + a^i) &= 2^{t(n,j)+1} + 2^{m(n,i)}, \\
f(a^n + a^j) &= 2^{t(n,j)+1}, \\
f(a^n + a) &= 2^{t(n,1)} + 2^{m(n,1)},
\end{aligned}$$

for all large n.. Now we must have $t(n,1) = t(n,j)$. Again using the same reasoning as above we will get $m(n,1) = t(n,1)$ which will mean $f(a^n + a^j) = f(a^n + a)$ for all large n. Contradiction! Hence the claim. ☐

Let us introduce some notation: Let $i \in \mathbb{N}$ and define t_n and $A(n,i)$ such that

$$f(a^n + a^i) = b^{t_n} + b^{A(n,i)},$$

for all large n (here we are using the previous claim, and t_n is independent of i for small i). Let $f(x) = x^d(xg(x) + c)$ where $c \neq 0$ and $g \in \mathbb{Q}[x]$.

Claim 2. Let p be a prime such that $p \mid b$ then $p \mid a$.

Proof. Assume that $\gcd(a,b) = 1$. Let r be some positive integer. Notice that $a^{c_1\phi(b^r)+d_1} + a^{c_2\phi(b^r)+d_2} \equiv a^{d_1} + a^{d_2} \pmod{b^r}$. Therefore if we take $c_1, c_2 \to \infty$ then using the previous claim, we get $b^r \mid f(a^{c_1\phi(b^r)+d_1} + a^{c_2\phi(b^r)+d_2}) \implies b^r \mid f(a^{d_1} + a^{d_2})$. Now taking r to be sufficiently large we get $f(a^{d_1} + a^{d_2})$, which means that $f \equiv 0$, this is a contradiction to our assumption that f is non-constant. □

Claim 3. Let $N \in \mathbb{N}$ be a constant. Then it follows that $\{A(n,i)\}_{i=1}^{N}$ forms an A.P. for large enough n and a is a power of b.

Proof. Let $p \mid \gcd(a,b)$ be a prime. Now consider

$$(a^n + a^i)^d((a^n + a^i)g(a^n + a^i) + c) = b^{t_n} + b^{A(n,i)}.$$

Taking ν_p of both sides and $n \to \infty$,

$$di\nu_p(a) + \nu_p(c) = A(n,i)\nu_p(b) \implies A(n,i) = \frac{id\nu_p(a) + \nu_p(c)}{\nu_p(b)}.$$

Hence the claim. Notice that the above also gives us that $\nu_p(b)(A(n,i+1) - A(n,i)) = d\nu_p(a)$, which means both a and b have the same set of prime divisors. Now if a prime q divides both a and b then by the same reasoning we have that

$$A(n,i) = \frac{id\nu_q(a) + \nu_q(c)}{\nu_q(b)} = \frac{id\nu_p(a) + \nu_p(c)}{\nu_p(b)},$$

taking $i = 1, 2$ we get that

$$\frac{\nu_q(a)}{\nu_q(b)} = \frac{\nu_p(a)}{\nu_p(b)} \implies a = b^r,$$

for some $r \in \mathbb{N}$. □

Finishing the problem is easy using the above claim.

9. Acknowledgements

I am very thankful to `Superguy`[2] for valuable suggestions and for being a problem resource.

[2] AoPS user : https://artofproblemsolving.com/community/user/388865