5 Problems 1 Solution : The Only Even Prime Number

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Under the title '5 problems 1 solution', I intend to discuss 5 problems, which can be solved using the same concept. I intend to keep the concept so simple that I expect we all know this concept. The interesting part is how such a simple concept can be used to solve various type of maths Olympiad problems.

Problem 1. Let p, q, r be distinct prime numbers. Prove that $p + q + r + pqr$ is a composite number.

(Berkeley Math circle, 2009)

Solution: Let $N = p + q + r + pqr$. We know that 2 is the only even prime number so, two cases arise:

Case I: p, q, r are all odd.

Then pqr is also odd. So, N is the sum of 4 odd numbers and hence even or a multiple of 2. Also, $N > 2$. So, N is composite.

Case II: One of p, q, r is 2 and the rest are odd. (Note that all the primes p, q, r are distinct. So, only one of them can be even.)

Let $p = 2$, and q, r are odd. So, $N = 2 + q + r + 2qr$.

Now, $q + r$ is even. So, is 2 and $2qr$. So, N is even and $N > 2$. So, N is a composite number.

Problem 2. Find all primes p, q such that $2p^3q^2 = 2(p + q)^2$.

(Junior Balkan Maths Olympaid, Shortlist – 2011)

Solution: We have

$$q^2 = 2p^3(p+q)^2. \tag{1}$$

We know that 2 is the only even prime number. From equation (1), we know that q is even. But q is also prime as given in the question. So $q = 2$. So, equation (1) becomes $4 = 2p^3 2(p+2)^2$, which implies

$$(p3)(p^2 + 2p + 2) = 0.$$

Hence $p = 3$, [since p is prime, so $p > 0$. Hence $p^2 + 2p + 2 \neq 0$.] So, $(p, q) = (3, 2)$.

Problem 3. Find all pairs of prime numbers (p, q) for which $7pq^2 + p = q^3 + 43p^3 + 1$.

(Dutch, 2015)

Solution: From the given equation we get

$$q^2(7p - q) = p(43p^2 - 1) + 1. \tag{2}$$

Now, if p and q are both odd, then LHS is even while RHS is odd, which is not possible. So, we know that we will use the concept that 2 is the only even prime number. Also, whether p is odd or even, the expression $p(43p^2 1)$ is always even and hence $p(43p^2 1) + 1$ is always odd. So, $q^2(7pq)$ is odd. Hence, q cannot be even; so, p must be even. But p is also prime, so $p = 2$. Using $p = 2$ in equation (2) we have,

$$14q^2 - q^3 = 343,$$

which implies that $q^3 = 7(2q^2 - 49)$. So, q is a multiple of 7. But q is prime. So $q = 7$, which satisfies equations (2). Hence $p = 2, q = 7$.

Problem 4. Find all triples (p, q, r) of primes such that $pq = r + 1$ and $2(p^2 + q^2) = r^2 + 1$.

(Regional Mathematical Olympiad, India – 2013)

Solution: We have $r^2 = 2(p^2 + q^2) - 1$. So, r is odd. Hence, $pq(= r + 1)$ is even. But both p and q are primes, so $p = 2$ or $q = 2$.

Suppose $p = 2$, then we have $r = 2q - 1$ and $r^2 = 2q^2 + 7$.

Using above two relations we get $q = 3$. So, $r = 2q1 = 5$.

Hence $(p, q, r) = (2, 3, 5)$ or $(3, 2, 5)$.

Problem 5. Let p and q be prime numbers. Show that $p^2 + q^2 + 2020$ is composite.

(Kosavo, 2020)

Solution: Here three cases arise:

Case I: p, q are both odd. Then $p^2 + q^2 + 2020$ is a multiple of 2 and hence composite.

Case II: p, q are both even. Then also $p^2 + q^2 + 2020$ is a multiple of 2 and hence composite.

Case III: one of p, q is odd and other is even. So, let $N = p^2 + q^2 + 2020$.

Suppose $p = 2$ and q is odd. Then $N = q^2 + 2024$. So, we need to prove that N is always composite.

Now, if $q = 3$, than $N = 9 + 2024 = 2033 = 19 \times 107$.

Let $q \neq 3$. We can write $N = (q1)(q + 1) + 2025$. Since, q is not a multiple of 3, so either $q1$ or $q + 1$ is a multiple of 3. Also, $2025 = 3 \times 675$. So, N is a multiple of 3 and hence composite.

Besides these 5 problems, there are many problems that can be solved using the concept that 2 is the only even prime number, of course in some of them some other concepts may also be required. The question given below are in this category that can be solved using "2 is the only even prime number" besides some other concept too.

1. Given x, y, z are primes such that $x^y + 1 = z$. Find x, y, z.

(Assam Mathematics Olympiad, 2015)

(Hints: $a^n + b^n$ is a multiple of $a + b$ whenever n is an odd positive integer.)

2. Find the triplets of primes (a, b, c) such that $ab8$ and $bc8$ are primes.

(Hintotusbashi University, Japan – 2014)

(Hints: a, b are both odd so, $ab8 = 2$.)

3. Find all prime numbers p, q, r satisfying the equation $p^4 + 2p + q^4 + q^2 = r^2 + 4q^3 + 1$.

(Turkey Junior National Olympiad, 2013)

4. Find all prime numbers p, q and r such that $p > q > r$ and the numbers pq, pr and qr are also prime.

(Solvenia, 2010)

(Hints: Show that p and q are both odd and r is even and hence $pq = 2, r = 2$.)

5. There are three prime numbers. If the sum of their squares is 5070, then what is the product of these numbers?

(International Mathematics competition, 2007)

(Hints: One of the primes is obviously 2. Prove that amongst the remaining two primes one is 5.)