Questions and hints to the problems of Assam Mathematics Olympiad 2020

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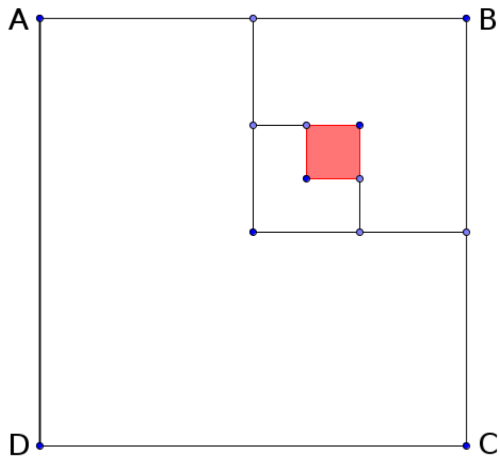
[Figures in the right margin indicate full marks for the question]

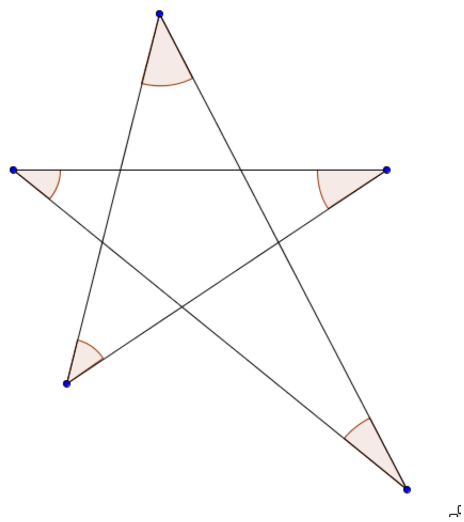
Like any other math problem, these problems can also be solved in several ways. We provide some hints here. Interested students can communicate their own solutions with us too.

(1) If the area of the square ABCD is 4544 sq.cm., find the area of the shaded region. The dots on the sides represent mid-points. 2



Solution. Shaded area $= \frac{1}{64} \times 4544 = 71.$ □

(2) Draw a star with five corner vertices, not necessarily regular. Let S be the sum of the five corner angles (shaded). What is the value of S/4 ? 3



Solution. Just use angle sum property of triangles to show that $S = 180°$. Hence, $\frac{S}{4} = 45°$. □

(3) Given that the equation $x^2 - ax + 12 = 0$ has positive integer roots, find the sum of all possible values of a. 3

Solution. Let α and β be the roots. Then, $\alpha\beta = 12$ and $\alpha + \beta = a$. Since α and β are positive integers, so we have the following possibilities only :
$\alpha\beta = 1 \times 12 = 2 \times 6 = 3 \times 4$
All possible values of a are 13, 8 and 7. Hence, required sum is 28. □

(4) Find the face value of the digit x such that the number $136119102856851x44$ is divisible by 17. 3

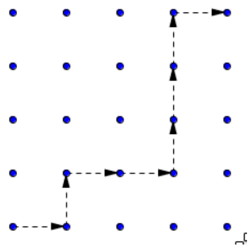
Solution. Observe that the number can be grouped to a few multiples of 17.

$$\begin{aligned}
&136119102856851x44\\
=&136 \times 10^{15} + 119 \times 10^{12} + 102 \times 10^{9} + 85 \times 10^{7} + 68 \times 10^{5} + 51 \times 10^{3} + 100x + 44\\
\equiv&100x + 44 \pmod{17}\\
\equiv&15x + 10 \pmod{17}\\
\equiv&-2x + 10 \pmod{17}
\end{aligned}$$

If the number is divisible by 17, then $-2x + 10 \equiv 0 \pmod{17}$ and since gcd(2,17)=1, so $x \equiv 5 \pmod{17}$. But x is a digit. So, $x = 5$. □

(5) How many paths can you draw from the point (0,0) to the point (4,4) in the Cartesian plane if you are allowed to move only north or east and each step is restricted to be of length one unit only? 3

Solution. One such path is shown below :



Under the given restrictions, in order to go from (0,0) to (4,4) we need exactly 4 north steps and exactly 4 east steps in any possible order. So, the number of paths is equal to the number of permutations of NNNNEEEE which is
$\frac{8!}{4! \times 4!} = \frac{8 \times 7 \times 6 \times 5}{4 \times 3 \times 2} = 70.$ □

(6) Let $P(x)$ be a polynomial with positive integer coefficients such that the product of all the coefficients (including the constant term) is 41. The remainder when $P(x)$ is divided by $x - 1$ is 78. What is the degree of $P(x)$? 4

Solution. Let degree of $P(x)$ be n. Since all the coefficients are positive integers and their product is 41 which is prime, so all coefficients must be 1 except one of the coefficients which must be 41. The remainder when $P(x)$ is divided by $x - 1$ is $P(1)$. Now, $P(1)$ is the sum of n ones and 41. So, $P(1) = n + 41$. Given, $P(1) = 78$. So, $n + 41 = 78$ and this gives $n = 37$. □

(7) You have Rs 172 and you want to spend this exact amount in buying pens and pencils worth Rs 11 and Rs 7 each respectively. How many ways can you make the purchase ? 4

Solution. Let x pens and y pencils be bought. Then, we need to find the non-negative integer solutions of $11x + 7y = 172$.

$$\begin{aligned} & 11x + 7y = 172 \\ \Rightarrow & 11x + 7y = 165 + 7 \\ \Rightarrow & 11x + 7y = 11 \times 15 + 7 \\ \Rightarrow & 11(x - 15) = 7(1 - y) \end{aligned}$$

So, 11 divides $7(1 - y)$. But gcd(11,7)=1. So, 11 divides $1 - y$ and so divides $y - 1$. Since y is non-negative, so $y - 1 = 0, 11, 22, 33, \ldots$. Out of these it is seen that the only possible solutions are $y = 1, x = 15$, $y = 12, x = 8$ and $y = 23, x = 1$. So there are 3 possible ways of purchase. □

(8) Suppose that the least values obtained by the quadratics $x^2 + 3x + 19$ and $x^2 - 7x + 3$ (as x varies over the set of real numbers) are a and b. What is a-b ? 4

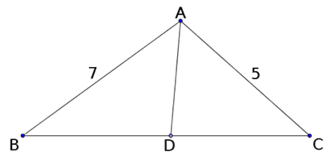
Solution.

$$\begin{aligned} x^2+3x+19 &= \left(x+\frac{3}{2}\right)^2+19-\frac{9}{4} \\ &= \left(x+\frac{3}{2}\right)^2+\frac{67}{4} \\ &\geq \frac{67}{4} \\ x^2-7x+3 &= \left(x-\frac{7}{2}\right)^2+3-\frac{49}{4} \\ &= \left(x-\frac{7}{2}\right)^2-\frac{37}{4} \\ &\geq -\frac{37}{4} \end{aligned}$$

Thus, $a=\frac{67}{4}$ and $b=-\frac{37}{4}$ and $a-b=26$. □

(9) AD is the bisector of angle BAC in a triangle ABC where D is a point on BC . Given that AB=7cm, AC=5cm and area(ABC)=156 sq.cm, what is the area of triangle ABD ? 5

Solution. Since AD bisects ∠ BAC, so $\frac{BD}{DC}=\frac{AB}{AC}=\frac{7}{5}$.



Triangles ABD and ADC have the same height. So, ratio of their areas is same as the ratio of their bases.

So, areas of triangles ABD and ADC are in the ratio 7:5 and hence area of triangle ABD is $\frac{7}{12}\times 156=91$. □

(10) A magic room contains 343 magic wands of 7 different colours. You are given a boon to ask for any number of wands as you wish. What is the minimum number of wands that you should wish for so that you get at least 14 wands of the same colour ? 5

Proof. In the worst case scenario, it may happen that there are 13 wands of each of the 7 colours. So, even if we wish for $7\times 13=91$ wands, we cannot guarantee that we will get at least 14 wands of the same colour. Anything less than 91 also doesn't guarantee this. Thus, we must wish for at least 92 wands. □

(11) The Fibonacci numbers are defined by the recurrence formula
$F(n)=F(n-1)+F(n-2)$ with $F(1)=1$ and $F(2)=1$.
What is the smallest odd prime factor of F(2020)? 5

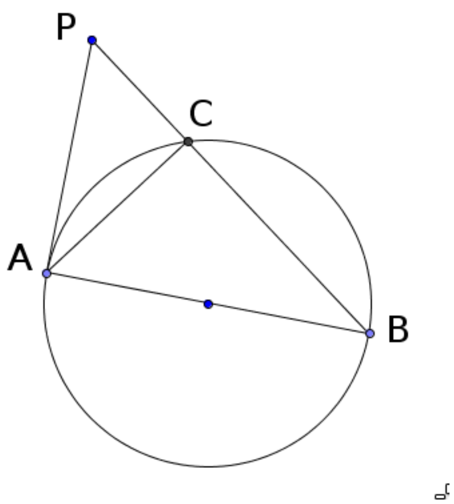
Solution. Begin by observing the first few Fibonacci numbers. F(1)=1, F(2)=1, F(3)=1+1=2, F(4)=2+1=3, F(5)=3+2=5, F(6)=5+3=8, F(7)=8+5=13, F(8)=13+8=21. The smallest odd prime factor of F(4) and F(8) is 3. Let us work modulo 3 to see if we get any clue. Observe that the recurrence formula gives
F(1)≡ 1 (mod 3), F(2)≡ 1 (mod 3), F(3)≡ 2 (mod 3), F(4)≡ 0 (mod 3)
F(5)≡ 2 (mod 3), F(6)≡ 2 (mod 3), F(7)≡ 1 (mod 3), F(8)≡ 0 (mod 3)

Try to prove that if n is a multiple of 4, F(n) is divisible by 3. Hence, the smallest odd prime factor of F(2020) is 3. □

(12) Let x be the number of ways of distributing 4 distinguishable balls into 2 distinguishable boxes and y be the number of ways to distribute 4 distinguishable balls in 9 distinguishable boxes such that each box gets at most one ball. What is the value of $y/3x$? 5

Solution. Since the boxes and balls are distinguishable, so 1st ball can be placed in 2 ways, 2nd in 2 ways, 3rd in 2 ways and 4th in 2 ways. By multiplication rule of counting, $x = 2 \times 2 \times 2 \times 2 = 2^4$. In the second case, each box can contain at most one ball. So, 1st ball can be placed in 9 ways, 2nd in 8 ways, 3rd in 7 ways and 4th in 6 ways. So, $y = 9 \times 8 \times 7 \times 6 = 2^4 \times 3 \times 63$. Hence, $y/3x = 63$. □

(13) In the given figure, AB is a diameter of the circle. PA is the tangent at A. If the radius is 4 cm and length of tangent is 6 cm, what is the value of 5.PC ? 5



Solution. Radius is perpendicular to the tangent at the point of contact. So, in triangle PAB, $\angle PAB = 90°$. By Pythagoras Theorem, $PB^2 = PA^2 + AB^2 = 6^2 + 8^2 = 100$. So, $PB = 10$. Now, $PC \times PB = PA^2$. So, $10 \times PC = 36 \Rightarrow 5 \cdot PC = 18$. □

(14) You are given the task of drawing triangles of perimeter 17 units but with the restriction that the sides must be integers. What is the maximum number of triangles that you can draw? 5

Solution. Let a, b, c be the sides. Then, $a + b + c = 17$. Now, sum of two sides of a triangle is greater than the third side. So,
$c < a + b \Rightarrow 2c < a + b + c \Rightarrow c < \frac{17}{2}$. So, $c \le 8$.
Similarly, $a \le 8$ and $b \le 8$. We can assume $a \le b \le c \le 8$. Since the sides are integers, so we have the following 8 possibilities :

a	b	c
1	8	8
2	7	8
3	6	8
3	7	7
4	5	8
4	6	7
5	5	7
5	6	6

□

(15) Let x and y be digits. Six digit numbers of the form $xyyxxy$ are constructed. How many of these are divisible by 7 ? 7

Solution.

$$\begin{aligned} xyyxxy &= 100110x + 11001y \\ &\equiv 3x + 4y \pmod 7 \\ &\equiv 3x - 3y \pmod 7 \end{aligned}$$

Since the number is divisible by 7, so $3(x - y) \equiv 0 \pmod 7$. But, gcd(3,7)=1. Hence, $x \equiv y \pmod 7$. One possibility is $x = y$. In this case, there are 9 numbers, 111111, 222222, 333333, ..., 999999. Since x and y are digits, so the other possibilities are:

$$\begin{aligned} x = 9, y = 2 &\text{ i.e. } 922992 \\ x = 8, y = 1 &\text{ i.e. } 811881 \\ x = 7, y = 0 &\text{ i.e. } 700770 \\ x = 2, y = 9 &\text{ i.e. } 299229 \\ x = 1, y = 8 &\text{ i.e. } 188118 \end{aligned}$$

So, there are 14 such numbers. □

(16) Find the number of integers between 1 and 200, both inclusive, that are not divisible by 3,5 and 8. 7

Solution. Let S be the set of integers between 1 and 200. Let A, B, C be subsets of S containing numbers divisible by 3, 5, 8 respectively. Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. Then,

$|A| = \left\lfloor \frac{200}{3} \right\rfloor = 66, |B| = \left\lfloor \frac{200}{5} \right\rfloor = 40, |C| = \left\lfloor \frac{200}{8} \right\rfloor = 25$
$|A \cap B| = \left\lfloor \frac{200}{15} \right\rfloor = 13, |B \cap C| = \left\lfloor \frac{200}{40} \right\rfloor = 5, |C \cap A| = \left\lfloor \frac{200}{24} \right\rfloor = 8$
$|A \cap B \cap C| = \left\lfloor \frac{200}{120} \right\rfloor = 1.$
By principle of inclusion-exclusion,

$$\begin{aligned}|A^c \cap B^c \cap C^c| &= |S| - (|A| + |B| + |C|) + (|A \cap B| + |B \cap C| + |C \cap A|) - |A \cap B \cap C| \\ &= 200 - (66 + 40 + 25) + (13 + 5 + 8) - 1 \\ &= 94\end{aligned}$$

□

(17) Find the value of $7x + 9y$ if

$$\sum_{k=1}^{n} k \binom{n}{k}^2 = n \binom{xn - y}{n - 1}$$

where $\binom{n}{k}$ represents ${}^nC_k$ i.e. the number of combinations of n things taken k at a time. 7

Solution. Let S be the sum in LHS. Since $\binom{n}{k} = \binom{n}{n-k}$, we observe that

$$\begin{aligned}
S &= 0 \cdot \binom{n}{0}^2 + 1 \cdot \binom{n}{1}^2 + 2 \cdot \binom{n}{2}^2 + 3 \cdot \binom{n}{3}^2 + \ldots + (n-1) \cdot \binom{n}{n-1}^2 + n \cdot \binom{n}{n}^2 \\
\Rightarrow S &= n \cdot \binom{n}{n}^2 + (n-1) \cdot \binom{n}{n-1}^2 + (n-2) \cdot \binom{n}{n-2}^2 + \ldots + 1 \cdot \binom{n}{1}^2 + 0 \cdot \binom{n}{0}^2 \\
\Rightarrow S &= n \cdot \binom{n}{0}^2 + (n-1) \cdot \binom{n}{1}^2 + (n-2) \cdot \binom{n}{2}^2 + (n-3) \cdot \binom{n}{3}^2 + \ldots + 1 \cdot \binom{n}{n-1}^2 + 0 \cdot \binom{n}{n}^2 \\
\Rightarrow 2S &= n \left\{ \binom{n}{0}^2 + \cdot \binom{n}{1}^2 + \binom{n}{2}^2 + \binom{n}{3}^2 + \ldots + \binom{n}{n-1}^2 + \binom{n}{n}^2 \right\} \\
\Rightarrow 2S &= n \binom{2n}{n} \qquad \text{Standard result} \\
\Rightarrow 2S &= n \frac{(2n)!}{n!n!} \\
\Rightarrow S &= n \frac{(2n-1)!}{(n-1)!n!} \\
\Rightarrow S &= n \binom{2n-1}{n-1}
\end{aligned}$$

Thus, $x = 2$ and $y = 1$. Hence, $7x + 9y = 23$. □

(18) How many positive integers have the property that sum of the positive square roots of the integer and its successor is less than 19 ? 7

Solution. Let n be a positive integer such that $\sqrt{n} + \sqrt{n+1} < 19$.

Observe that if $\sqrt{n} > 10$, then $\sqrt{n} + \sqrt{n+1} > 20$. In fact, if $\sqrt{n} \geq 9.5$, then also $\sqrt{n} + \sqrt{n+1} > 19$. Thus, we must have $\sqrt{n} < 9.5$, so that $n < 90.25$. This gives, $n \leq 90$. It can be shown that $n = 90$ doesn't satisfy the given condition but $n \leq 89$ satisfies it. □

(19) Let x be an integer such that $5x^2 - 98x$ is 57 less than a power of prime. Find the largest such prime.8

Proof. As per question,

$$
\begin{aligned}
&5x^2 - 98x = p^n - 57 \\
\Rightarrow &5x^2 - 98x + 57 = p^n \\
\Rightarrow &5x^2 - 95x - 3x + 57 = p^n \\
\Rightarrow &(5x-3)(x-19) = p^n
\end{aligned}
$$

Since $5x - 3$ and $x - 19$ are integers and p is a prime, there are the following cases :

(a) $5x - 3 = 1, x - 19 = p^n$ (Not possible. Why?)
(b) $5x - 3 = -1, x - 19 = -p^n$ (Not possible. Why?)
(c) $5x - 3 = p^n, x - 19 = 1$ which gives $p = 97$ and $n = 1$.
(d) $5x - 3 = -p^n, x - 19 = -1$ (Not possible. Why?)
(e) p divides both $5x - 3$ and $x - 19$. In this case, by property of divisibility, $p|(5x-3) - 5(x-19)$ so that $p|92$. But $92 = 2 \times 2 \times 23$. So, $p = 2$ or 23.

Hence, largest possible prime is 97. □

(20) Three friends play a game where each of them chooses a positive integer and they multiply the three integers thus obtained. The choice is said to be successful if the product is 5252. How many successful choices are possible ? 8

Solution. Let x, y, z be the numbers chosen by the friends. The problem is to find the number of ordered triplets of positive integers (x, y, z) such that $xyz = 5252 = 2^2 \times 13 \times 101$. Since x, y, z are factors of 5252, so we can write

$$
\begin{aligned}
x &= 2^{a_1} \times \mathbf{13}^{b_1} \times 101^{c_1} \\
y &= 2^{a_2} \times \mathbf{13}^{b_2} \times 101^{c_2} \\
z &= 2^{a_3} \times \mathbf{13}^{b_3} \times 101^{c_3}
\end{aligned}
$$

Thanks to Supratik Chattopadhyay for correcting the typo. 13 was typed as 5.

where

$$a_1 + a_2 + a_3 = 2 \tag{1}$$

$$b_1 + b_2 + b_3 = 1 \tag{2}$$

$$c_1 + c_2 + c_3 = 1 \tag{3}$$

Number of non-negative integer solutions of (1) is 6, of (2) is 3 and of (3) is 3. Thus, number of ordered triplets (x, y, z) is $6 \times 3 \times 3$=54. □