# Problem Section 1

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We are starting a new section from this issue, which will have problems without solutions. We ask the solutions from the readers which we will publish in the subsequent issues. All solutions should preferably be typed in LaTeX and emailed to the editor. If you would like to propose problems for this section then please send your problems (with solutions) to the above mentioned email address, preferably typed in LaTeX. Each problem or solution should be typed on separate sheets. Solutions to problems in this issue must be received by *20 March, 2021.* If a problem is not original, the proposer should inform the editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in Ganit Bikash. Solvers are asked to include references for any non-trivial results they use in their solutions.

**Problem 1.** *Proposed by Anupam Saikia (Indian Institute of Technology Guwahati).*

A composite number $m$ is called a pseudo-prime to the base $a$ if $a^{m-1} \equiv 1 \pmod{m}$. Show that if $p$ is an odd prime not dividing $a^2-1$, then $m = \dfrac{a^{2p}-1}{a^2-1}$ is a pseudo-prime to the base $a$. (*This shows that there are infinitely many pseudo-primes to any base* $a > 1$.)

**Problem 2.** *Proposed by Manjil P. Saikia (Cardiff University).*

A partition of $n$ is a sequence of non-increasing integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ such that $\sum_{i=1}^{k} \lambda_i = n$. The integers $\lambda_i$ are called parts of the partition $\lambda$ and $p(n)$ denotes the total number of partitions of $n$. For instance $5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1$ and $1+1+1+1+1$ are all the partitions of 5, so $p(5) = 7$. Let us define the following partition statistics:

- $a_k(n)$ to be the sum of the parts which are divisible by $k$ counted without multiplicity in all the partitions of $n$,

- $a_{k,p}(n)$ to be the sum of the parts which are congruent to $p \pmod{k}$ counted without multiplicity in all the partitions of $n$, and
- $b_k(n)$ to be the sum of the distinct parts that appear at least $k$ times in all the partitions of $n$.

Then prove that

(i) $a_k(n) = kb_k(n)$, and

(ii) $a_{3,1}(n) = 2b_3(n-1) + b_3(n+2)$.

**Problem 3.** *Proposed by B. Sury (Indian Statistical Institute, Bangalore).*

Let $f : \{(x,y) : x, y \in \mathbf{R}, xy \neq 0\} \to \{t \in \mathbf{R} : t > 0\}$ be a function satisfying the conditions

$$f(xy, z) = f(x, z)f(y, z) \ \forall\ x, y, z \neq 0,$$

$$f(x, yz) = f(x, y)f(x, z);$$

and

$$f(x, 1-x) = 1 \ \forall\ x \neq 0, 1.$$

Prove:

(a) $f(x,x) = f(x,-x) = 1$ for all $x \neq 0$;

(b) $f(x,y)f(y,x) = 1$ for all $x, y \neq 0$.

"I've noticed that there are three ways in which scientists grow old. Some stop doing research and drop out to do other things. The second group gets bored and overreaches by going into new fields where they don't have the background — Linus Pauling, William Shockley and Fred Hoyle come to mind there.
A third way is just to keep on doing what you're good at and accept that you may not be scaling more than a plateau."

**– Martin Rees**

*Former President of the Royal Society (2005 to 2010).*