The Remarkable Sequence $1, 2, 7, 42, 429, 7436, \cdots$

Dr. Manjil P. Saikia

School of Mathematics, Cardiff University, CF24 4AG, UK

E-mail: manjil@saikia.in

Abstract. The sequence of numbers starting with $1, 2, 7, 42, 429, 7436, \cdots$ counts several seemingly different combinatorial objects, which have kept mathematicians busy for the past four decades in their effort to understand this sequence better. We give a brief historical overview of these efforts related to only one class of objects that the sequence counts, without claiming to be comprehensive. We assume basic school level knowledge of linear algebra.

The sequence in the title of this article is given by the following 'nice' formula

$$\frac{1!4!7!\cdots(3n-2)!}{n!(n+1)!\cdots(2n-1)!}$$

or, in a more compact product notation

$$\prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}.$$

Here $k!$ denotes the product $1 \cdot 2 \cdot 3 \cdots k$. This formula was conjectured by Mills, Robbins and Rumsey [MRR83] to count what are called *alternating sign matrices* (ASMs). By a 'nice' formula like the one above, we combinatorialists usually mean formulas that can be written as products of factorials or sums of products of factorials[1]

An **alternating sign matrix** (ASM) of size n is an $n \times n$ matrix with entries in the set $\{0, 1, -1\}$ such that

- all row and column sums are equal to 1,

[1] There are several other classes of nice formulas that we deal with, but for the purposes of this article, we do not mention them.

- and the non-zero entries alternate in each row and column.

For instance, there are 7 ASMs of order 3, these are the six permutation matrices

$$\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} ; \begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix} ; \begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{pmatrix} ; \begin{pmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{pmatrix} ; \begin{pmatrix}0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}$$

and the matrix

$$\begin{pmatrix}0 & 1 & 0\\ 1 & -1 & 1\\ 0 & 1 & 0\end{pmatrix}.$$

From the definition of ASMs, it is clear that all permutation matrices are ASMs.

But how and why did they decide to study these matrices? To answer that question, we have to start with the familiar determinants. For the matrix

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}$$

the determinant is

$$ad - bc.$$

For the matrix

$$\begin{pmatrix}a & b & c\\ d & e & f\\ g & h & i\end{pmatrix}$$

the determinant is

$$aei + bfg + cdh - ceg - bdi - afh.$$

More generally, for an $n \times n$ matrix, A with entries $a_{i,j}$ $(1 \leq i, j \leq n)$, the determinant of A is defined as

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}.$$

Determinants appear in all sorts of contexts in combinatorics and they are widely studied. We say that the determinant is of order n if the matrix is of order n^2.

For the remainder of the article, A will always denote an $n \times n$ matrix unless otherwise stated. Given a matrix A, we let A^i_j denote the matrix that remains when the ith row and jth column of A are deleted. If we remove more than one row or column, then the indices corresponding to those are added to the super- and sub- scripts. Determinants can be calculated in an algorithmic way using the following famous result called the **Desnanot-Jacobi identity**.

Theorem 1 (Desnanot-Jacobi adjoint matrix theorem)**.** *If A is an $n \times n$ matrix, then*

$$\det(A)\det(A^{1,n}_{1,n}) = \det(A^1_1)\det(A^n_n) - \det(A^1_n)\det(A^n_1)$$

or

$$\det(A) = \frac{1}{\det(A^{1,n}_{1,n})} \times \det\begin{pmatrix} \det(A^1_1) & \det(A^1_n) \\ \det(A^n_1) & \det(A^n_n) \end{pmatrix}.$$

Reverend Charles L. Dodgson, better known by his pen name of Lewis Carroll used Desnanot-Jacobi theorem to give an algorithm for evaluating determinants in terms of 2×2 determinants. For instance, we get

$$\det\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} = \frac{1}{a_{2,2}} \times \det\begin{pmatrix} \det\begin{pmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{pmatrix} & \det\begin{pmatrix} a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{pmatrix} \\ \det\begin{pmatrix} a_{1,2} & a_{1,3} \\ a_{2,2} & a_{2,3} \end{pmatrix} & \det\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \end{pmatrix}.$$

This reduces the calculation of an order 3 determinant to calculating four order 2 determinants. We can go one step further

$$\det\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{pmatrix} = \frac{1}{\det\begin{pmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{pmatrix}} \times \det\begin{pmatrix} \det\begin{pmatrix} a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,2} & a_{4,3} & a_{4,4} \end{pmatrix} & \det\begin{pmatrix} a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \\ a_{4,1} & a_{4,2} & a_{4,3} \end{pmatrix} \\ \det\begin{pmatrix} a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,2} & a_{3,3} & a_{3,4} \end{pmatrix} & \det\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \end{pmatrix}.$$

[2] This is an abuse of notation, but we can live with this.

So, we have now rediued the calculation of an order 4 determinant to calculating four order 3 determinants and one order 2 determinant. One can keep on going in this way to calculate higher order determinats.

In the 1980s, Robbins and Rumsey looked at a generalization of the 2×2 determinant, which they called the λ-determinant. They defined

$$\det_l \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} = a_{1,1}a_{2,2} + \lambda a_{2,1}a_{1,2}.$$

Using the previous observations, they generalized it to an $n \times n$ determinant using the algorithmic way. They wanted to get a closed form expression for the value of this λ-determinant. Their main result in this direction was the following result.

Theorem 2 (Robbins-Rumsey)**.** *Let A be an $n \times n$ matrix with entries $a_{i,j}$, $\mathcal{A}_n$ be the set of all ASMs, $\mathcal{I}(B)$ be the inversion number of B and $\mathcal{N}(B)$ be the number of -1's in B. Then*

$$\det_l(A) = \sum_{B \in \mathcal{A}_n} \lambda^{\mathcal{I}(B)} (1 + \lambda^{-1})^{\mathcal{N}(B)} \prod_{i,j=1}^{n} a_{i,j}^{B_{i,j}}.$$

This was the first appearance of an ASM in the literature.

We briefly mention what the inversion number of an ASM is, which appears in the above theorem. An easy way to calculate the inversion number is to take products of all pairs of entries for which one of them lies to the right and above the other, and then adding them all up. For instance we look at the following order 5 ASM:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

There are seven pairs here whose product is $+1$ and two pairs whose product is -1. So the inversion number is 5. This trick will also work for calculating the inversion number of a permutation matrix.

But how does one get to the formula for the number of ASMs from these observations? Let us look at the above ASM. The first non-trivial observation that we can make is that there can be only one 1 in the top row (or, first column). This is easy to check (why?). Let $A_{n,k}$ be the number of $n \times n$ ASMs with a 1 at the top of the kth column. Some thought will give us, $A_{n,k} = A_{n,n+1-k}$ (symmetry). Further, if A_n is the number of $n \times n$ ASMs, then $A_{n,1} = A_{n,n} = A_{n-1}$. This allows one to check small values to get a formula. Mills, Robbins and Rumsey did exactly that.

They first conjectured the relation

$$\frac{A_{n,k}}{A_{n,k+1}} = \frac{k(2n-k-1)}{(n-k)(n+k-1)}.$$

This means that the $A_{n.k}$'s are uniquely determined by the $A_{n,k-1}$'s when $k > 1$ and by $A_{n,1} = \sum_{k=1}^{n-1} A_{n-1,k}$. The above conjecture can be reformulated as

$$A_{n,k} = \binom{n+k-2}{k-1} \frac{(2n-k-1)!}{(n-k)!} \prod_{j=0}^{n-2} \frac{(3j+1)!}{(n+j)!}.$$

It would be a nice exercise to prove this reformulation from our observations. From here, knowing that $A_n = A_{n+1,1}$ allows one to conjecture the formula in the first page

$$A_n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}.$$

.

What about a proof? Doron Zeilberger [Zei96a] succeeded in proving this formula (called the ASM conjecture) in 1996 using constant term identities[3]. The proof ran for 84 pages and involved a team of referees to check all the details, and remains to this day a unique paper in that sense. Shortly after, Greg Kuperberg [Kup96] gave an alternate proof by exploiting a connection between ASMs and statistical physics. It turned out that physicists have been studying ASMs under a different guise since a long time. No one had made the connection before Kuperberg[4].

The formula for $A_{n,k}$ given above was also proven by Zeilberger [Zei96b] using the techniques used by Kuperberg. Such type of formulas are called refined enumeration results, because we are enumerating a class of objects with a refinement, in this case the position of the unique 1 in the first row. So, we see that a very simple object such as the ASM took a long and sustained effort before it could be counted. And when it was counted, there were two different proofs in quick succession. The beauty of a formula lies in its ability to attract such an outcome.

Does the story end here? Of course not, otherwise we would not have been talking about these objects. In the late 1980's Richard Stanley [Sta86] suggested the study of various symmetry classes of ASMs; this let Robbins to conjecture formulas for many of these classes. It turned out to be as difficult as enumerating ASMs, and this study was only recently completed in 2016.

The dihedral group of symmetries D_4 acts naturally on an ASM. This gives rise to the following symmetry classes (the people who proved the corresponding enumeration formula is given in parentheses):

- Vertically Symmetric ASMs: $a_{i,j} = a_{i,n+1-j}$, n odd (Kuperberg 2002)
- Half-turn Symmetric ASMs: $a_{i,j} = a_{n+1-i,n+1-j}$, n odd (Razumov-Stroganov 2005), n even (Kuperberg 2002)
- Diagonally Symmetric ASMs: $a_{i,j} = a_{j,i}$, no 'nice' formula

[3] He proved a more stronger result, but we do not discuss this here.
[4] Again, we do not discuss this here, maybe we will discuss in the next edition of this publication.

- Quarter-turn Symmetric ASMs: $a_{i,j} = a_{j,n+1-i}$, n odd (Razumov-Stroganov 2005), n even (Kuperberg 2002)
- Horizontally and vertically Symmetric ASMs: $a_{i,j} = a_{i,n+1-j} = a_{n+1-i,j}$, n odd (Okada 2004)
- Diagonally and Antidiagonally Symmetric ASMs: $a_{i,j} = a_{j,i} = a_{n+1-j,n+1-i}$, n odd (Behrend-Fischer-Konvalinka 2017)
- All symmetries: $a_{i,j} = a_{j,i} = a_{i,n+1-j}$, no 'nice' formula.

For instance, a vertically symmetric ASM of order 7 is the following

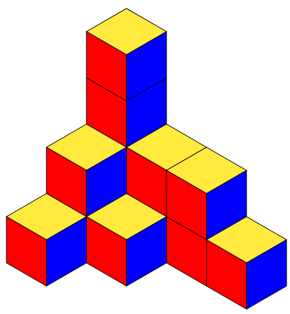
$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Are ASMs worth studying only because they are difficult to enumerate? Again the answer is an emphatic NO. They are intimately related to other objects that combinatorialists study such as plane partitions, descending plane partitions, non-intersecting lattice paths, etc. It would go beyond the scope of this article to discuss them in detail. We close this article with a tantalizing open problem related to ASMs and plane partitions.

A **plane partition** in an $a \times b \times c$ box is a subset

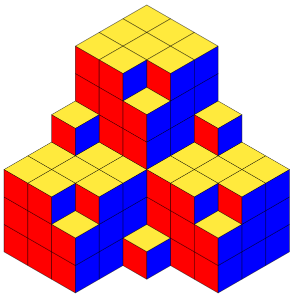
$$PP \subset \{1, 2, \cdots, a\} \times \{1, 2, \cdots, b\} \times \{1, 2, \cdots, c\}$$

with $(i', j', k') \in PP$ if $(i, j, k) \in PP$ and $(i', j', k') \leq (i, j, k)$. It is easy to visualize a plane partition, although an equivalent definition exists in terms of arrays of numbers with certain monotonicity conditions. Below we can 'see' a plane partition in an $3 \times 4 \times 4$ box.



Like ASMs, combinatorialists are also interested in symmetry classes of plane partitions. One of these symmetry classes is the class of **totally symmetric self-complementary plane**

partitions. If a plane partition has all the symmetries (that is, $(i, j, k) \in PP$ if and only if all six permutations of (i, j, k) are also in PP) and is its own complement (that is, if $(i, j, k) \in PP$ then $(2n + 1 - i, 2n + 1 - j, 2n + 1 - k) \notin PP$), then it is called *totally symmetric self-complementary plane partitions*(TSSCPP). An example of such a TSSCPP is given below.



The class of TSSCPPs inside a $2n \times 2n \times 2n$ box are known to be equinumerous with $n \times n$ ASMs. However, a bijective proof of this result is still not known and is considered to be one of the most important open problems in all of combinatorics.

There are many things that can be said about ASMs other than the ones discussed here. Instead of hearing it from this author, we refer the interested reader to the beautiful the book of Bressoud [Bre99] which discusses all the developments until the year 1999. The book also contains a wealth of information about other aspects of combinatorics and is strongly recommended by this author. For a brief overview of where things stand at the present moment, the recent articles of Behrend, Fischer and Konvalinka [BFK17] and that of Fischer and this author [FS21] contains useful surveys of known results until 2016 and 2019 respectively.

[BFK17] Roger E. Behrend, Ilse Fischer, and Matjaž Konvalinka. Diagonally and antidiagonally symmetric alternating sign matrices of odd order. *Adv. Math.*, 315:324–365, 2017.

[Bre99] David M. Bressoud. *Proofs and confirmations*. MAA Spectrum. Mathematical Association of America, Washington, DC; Cambridge University Press, Cambridge, 1999. The story of the alternating sign matrix conjecture.

[FS21] Ilse Fischer and Manjil P. Saikia. Refined enumeration of symmetry classes of alternating sign matrices. *J. Combin. Theory Ser. A*, 178, 105350, 51 pp., 2021.

[Kup96] Greg Kuperberg. Another proof of the alternating-sign matrix conjecture. *Internat. Math. Res. Notices*, (3):139–150, 1996.

[MRR83] W. H. Mills, David P. Robbins, and Howard Rumsey, Jr. Alternating sign matrices and descending plane partitions. *J. Combin. Theory Ser. A*, 34(3):340–359, 1983.

[Sta86] Richard P. Stanley. A baker's dozen of conjectures concerning plane partitions. In *Combinatoire énumérative (Montreal, Que., 1985/Quebec, Que., 1985)*, volume 1234 of *Lecture Notes in Math.*, pages 285–293. Springer, Berlin, 1986.

[Zei96a] Doron Zeilberger. Proof of the alternating sign matrix conjecture. *Electron. J. Combin.*, 3(2):Research Paper 13, approx. 84, 1996. The Foata Festschrift.

[Zei96b] Doron Zeilberger. Proof of the refined alternating sign matrix conjecture. *New York J. Math.*, 2:59–68, electronic, 1996.