# Assam Academy of Mathematics

## Assam Mathematics Olympiad 2022

## Category II (Classes VII - VIII)

Full marks : 100 Time : 3 hours

There are 18 questions. Questions 1 to 5 carry 2 marks each. Questions 6 to 13 carry 5 marks each. Questions 14 to 18 carry 10 marks each.

ইয়াত 18 টা প্ৰশ্ন আছে। 1 ৰ পৰা 5 লৈ প্ৰতিটো প্ৰশ্নত 2 নম্বৰকৈ আছে। 6 ৰ পৰা 13 লৈ প্ৰতিটো প্ৰশ্নত 5 নম্বৰকৈ আছে। আৰু 14 ৰ পৰা 18 লৈ প্ৰতিটো প্ৰশ্নত 10 নম্বৰকৈ আছে।

There may be various other ways of solutions than those shown here. Queries or suggestions regarding the solutions can be mailed to mail@aamonline.in

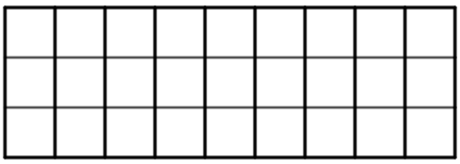
1. Let $a, b, c$ be digits such that the 3-digit numbers $abc, bca, cab$ are primes. Find the GCD and LCM of the 9-digit numbers $abcabcabc$, $bcabcabca$ and $cabcabcab$.

   ধৰা, $a, b, c$ তিনিটা দশমিক অংক যাতে তিনিটা অংকৰে গঠিত $abc, bca, cab$ সংখ্যা তিনিটা মৌলিক। ৯ টা অংকৰে গঠিত $abcabcabc$, $bcabcabca$ আৰু $cabcabcab$ সংখ্যা তিনিটাৰ গসাগু আৰু লসাগু নিৰ্ণয় কৰা।

   Ans : Observe that $abcabcabc = abc \times 1001001$, $bcabcabca = bca \times 1001001$ and $cabcabcab = cab \times 1001001$. Also, $abc, bca$ and $cab$ are primes. Thus, the GCD is $1001001$ and LCM is $abc \times bca \times cab \times 1001001$.

2. What is the total number of rectangles in the following grid?

   তলৰ জালিকাখনত থকা মুঠ আয়তক্ষেত্ৰৰ সংখ্যা কিমান?

   Ans : Each rectangle is formed by the intersection of two horizontal lines and two vertical lines. There are 4 horizontal lines and 10 vertical lines. Choice of two horizontal lines can be made in $\binom{4}{2} = C(4,2) = \dfrac{4!}{(4-2)!2!} = 6$ ways and for each choice of the horizontal lines, the choice of 2 vertical lines can be made in $C(10,2) = \dfrac{10!}{(10-2)!2!} = 45$ ways. So, total number of rectangles is $6 \times 45 = 270$.

3. If the radius of a circle is decreased by 10%, then what is the percentage decrease in its area?

যদি এটা বৃত্তৰ ব্যাসাৰ্ধ 10% হ্ৰাস কৰা হয়, তেন্তে ইয়াৰ ক্ষেত্ৰফল কিমান শতাংশ হ্ৰাস হ'ব?

Ans : Let original radius be $r$. New radius is 90% of $r$ i.e. $9r/10$. Area is $\pi(9r/10)^2 = \pi r^2 \times 81/100$. So, area is decreased by 19%.

4. Evaluate the following sum of 999 fractions:

তলত দিয়া 999 টা ভগ্নাংশৰ যোগফলটো নিৰ্ণয় কৰা:

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\ldots\ldots+\frac{1}{999\times 1000}.$$

Ans :

$$\begin{aligned}
&\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\ldots\ldots+\frac{1}{999\times 1000}\\
=&\frac{2-1}{1\times 2}+\frac{3-2}{2\times 3}+\frac{4-3}{3\times 4}+\ldots\ldots+\frac{1000-999}{999\times 1000}\\
=&\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\ldots\ldots+\left(\frac{1}{999}-\frac{1}{1000}\right)\\
=&1-\frac{1}{1000}\\
=&\frac{999}{1000}
\end{aligned}$$

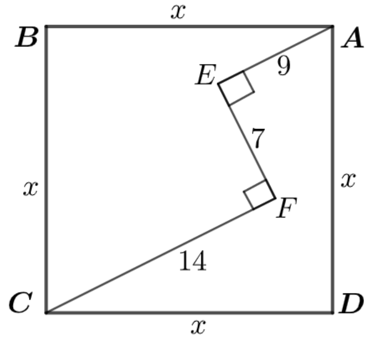
5. Let $\alpha$ be the least possible value of $x^2-2x+3$ and $\beta$ be the greatest possible value of $-x^2+2x+3$ as $x$ varies over real numbers. Find the value of $\alpha^3+\beta^3$.

ধৰা, $\alpha$ হৈছে $x^2-2x+3$ ৰ ন্যূনতম সম্ভাব্য মান আৰু $\beta$ হৈছে $-x^2+2x+3$ ৰ সৰ্বোচ্চ সম্ভাব্য মান, য'ত $x$ হৈছে কোনো বাস্তৱ সংখ্যা। $\alpha^3+\beta^3$ ৰ মান নিৰ্ণয় কৰা।

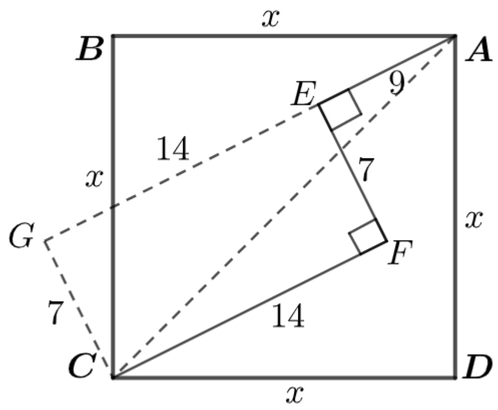
Ans : $x^2-2x+3=(x-1)^2+2\geq 2$ as the least value of $(x-1)^2$ is zero. So, $\alpha=2$. Also, $-x^2+2x+3=3-(x^2-2x)=3-(x-1)^2+1=4-(x-1)^2\leq 4$. So, $\beta=4$. Thus, $\alpha^3+\beta^3=8+64=72$.

6. In the figure $ABCD$ is a square of side $x$ units. Find $x$ if $AE\perp EF$, $EF\perp CF$, $AE=9$, $EF=7$ and $CF=14$.

তলৰ চিত্ৰত $ABCD$ হৈছে $x$ একক বাহু-দৈৰ্ঘ্যৰ এটা বৰ্গ। $x$ ৰ মান নিৰ্ণয় কৰা যদি $AE\perp EF$, $EF\perp CF$, $AE=9$, $EF=7$ আৰু $CF=14$।



Ans : Extend $AE$ to meet $CG$ parallel to $FE$ at $G$. In triangle, $ACG$, by Pythagoras theorem, $AC^2 = AG^2 + GC^2 = (9+14)^2 + 7^2 = 578$. Thus, $2x^2 = 578$ so that $x^2 = 289$ i.e. $x = 17$.




7. Justify if there exist non-zero rational numbers $a$ and $b$ satisfying the following equation.

তলৰ সমীকৰণটো মানি চলাকৈ অশূন্য পৰিমেয় সংখ্যা $a$ আৰু $b$ আছে নে নাই বিচাৰ কৰা।

$$a^2b^2(a^2b^2+4) = 2(a^6+b^6).$$

Ans :

$$\begin{aligned}
&a^2b^2(a^2b^2+4) = 2(a^6+b^6)\\
\Rightarrow &a^4b^4 + 4a^2b^2 - 2a^6 - 2b^6 = 0\\
\Rightarrow &a^4(b^4 - 2a^2) - 2b^2(b^4 - 2a^2) = 0\\
\Rightarrow &(a^4 - 2b^2)(b^4 - 2a^2) = 0\\
\Rightarrow &a^4 = 2b^2 \text{ or } b^4 = 2a^2\\
\Rightarrow &\frac{a^2}{b} = \pm\sqrt{2} or \frac{a^2}{b} = \pm\sqrt{2}
\end{aligned}$$

Thus, $a$ and $b$ cannot be both rational.

8. Find all ordered triples $(p, q, r)$ of primes such that $pq = r+1$ and $2(p^2+q^2) = r^2+1$.

মৌলিক সংখ্যাৰ আটাইবোৰ ক্ৰমিত ত্ৰয়ী $(p, q, r)$ নিৰ্ণয় কৰা যাতে $pq = r+1$ আৰু $2(p^2 + q^2) = r^2 + 1$।

Ans : $2(p^2+q^2) = r^2+1 \Rightarrow r^2 = 2(p^2+q^2) - 1$ which is odd. Hence, $r$ is an odd prime. So, $pq = r+1$ is even but 2 is the only even prime. Hence, $p = 2$

or $q = 2$. If $p = 2$, $r = 2q - 1$. So,

$$\begin{aligned}
&2(p^2 + q^2) = r^2 + 1\\
\Rightarrow &2(4 + q^2) = (2q - 1)^2 + 1\\
\Rightarrow &8 + 2q^2 = 4q^2 - 4q + 1 + 1\\
\Rightarrow &2q^2 - 4q - 6 = 0\\
\Rightarrow &q^2 - 2q - 3 = 0\\
\Rightarrow &(q - 3)(q + 1) = 0\\
\Rightarrow &q = 3 \text{ as } q \neq -1\\
\Rightarrow &r = 5
\end{aligned}$$

So, one ordered triple is $(2, 3, 5)$. The other is $(3, 2, 5)$.

9. Let $ABC$ be an acute angled triangle and $E, F$ be the feet of the altitudes through $B, C$ respectively such that $BE = \sqrt{3}AE$ and $CF = \sqrt{3}AF$. Prove that $CE - BF = \frac{3}{2}(AC - AB)$.

   ধৰা, $ABC$ এটা সুক্ষ্মকোণী ত্ৰিভুজ আৰু $E, F$ হৈছে ক্ৰমে $B, C$ ৰ পৰা টনা উন্নতি দুডালৰ পাদবিন্দু যাতে $BE = \sqrt{3}AE$ আৰু $CF = \sqrt{3}AF$। প্ৰমাণ কৰা যে $CE - BF = \frac{3}{2}(AC - AB)$।

   Ans : In triangle $ABE$, Pythagoras Theorem gives $AB^2 = AE^2 + BE^2 = AE^2 + 3AE^2 = 4AE^2$. So, $AE = AB/2$. Similarly, in triangle $AFC$, $AF = AC/2$. Thus,

$$\begin{aligned}
CE - BF &= AC - AE - (AB - AF)\\
&= AC - AB/2 - (AB - AC/2)\\
&= \frac{3}{2}(AC - AB)
\end{aligned}$$

10. Let $P(x)$ be a polynomial of degree 17 with positive integer coefficients. The product of the coefficients is 73. Find $P(1)$.

    $P(x)$ হৈছে ধনাত্মক পূৰ্ণসংখ্যা সহগ যুক্ত 17 মাত্ৰাৰ এটা বহুপদ ৰাশি। ইয়াৰ সহগবোৰৰ গুণফল হ'ল 73। $P(1)$ নিৰ্ণয় কৰা।

    Ans : Let $P(x) = a_0 + a_1x + a_2x^2 + \ldots\ldots + a_{17}x^{17}$. Since the coefficients are positive integers and $a_0a_1a_2 \cdots a_{17} = 73$, which is a prime number, so one of the coefficients is 73 and all the other seventeen coefficients are 1. Hence we have, $P(1) = a_0 + a_1 + a_2 + \ldots\ldots + a_{17}$ is the sum of 73 and seventeen 1's i.e. $P(1) = 17 + 73 = 90$.

11. A number $X$ is equal to the product of the first 10 multiples of $p$ where $p$ is a prime greater than 10. How many positive divisors does $X$ have?

    $X$ হৈছে $p$ ৰ প্ৰথম 10 টা গুণিতকৰ গুণফলৰ সমান, য'ত $p$ হৈছে 10 তকৈ ডাঙৰ মৌলিক সংখ্যা। $X$ ৰ কিমান সংখ্যক ধনাত্মক উৎপাদক আছে?

Ans : $X = p \times 2p \times 3p \times 4p \times 5p \times 6p \times 7p \times 8p \times 9p \times 10p = 2 \times 3 \times 4 \times 5 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 \times p^{10}$. Thus, $X = 2 \times 3 \times 2^2 \times 5 \times 2 \times 3 \times 7 \times 2^3 \times 3^2 \times 2 \times 5 \times p^{10} = 2^8 \times 3^4 \times 5^2 \times 7 \times p^{10}$. Hence the number of positive divisors is $(8+1)(4+1)(2+1)(1+1)(10+1) = 2970$.

12. Find the number of elements in the set

$$S = \{(x,y)|x,y \in N \text{ and } x^2 - y^2 = 5050\}$$

, where $N$ is the set of natural numbers.

$S = \{(x,y)|x,y \in N$ আৰু $x^2 - y^2 = 5050\}$ সংহতিটোৰ মৌলৰ সংখ্যা নিৰ্ণয় কৰা, য'ত $N$ স্বাভাৱিক সংখ্যাৰ সংহতিটো।

Ans :

$$\begin{aligned} &x^2 - y^2 = 5050 \\ \Rightarrow &(x-y)(x+y) = 2 \times 25 \times 101 \end{aligned}$$

The RHS has 2 as the only even divisor. So, when 5050 is written as a product of two numbers, one of the numbers is odd and the other is even. Thus, $(x-y)+(x+y) = 2x$ has to be odd if the above equality were to hold. But that cannot happen as $2x$ is even. Thus, there are no such pairs $(x,y)$ that lie in $S$. So, number of elements of $S$ is zero.

13. Brass is an alloy of copper and zinc. Bronze is an alloy containing 80% copper, 4% zinc and 16% tin. A mixed mass of bronze and brass is found to contain 74% copper, 16% zinc and 10% tin. Find the percentage composition of copper and zinc in brass.

পিতল হৈছে তাম আৰু জিংকৰ মিশ্ৰণ। ব্ৰঞ্জ হৈছে 80% তাম, 4% জিংক আৰু 16% টিনৰ মিশ্ৰণ। ব্ৰঞ্জ আৰু পিতলৰ এটা মিশ্ৰনত 74% তাম, 16% জিংক আৰু 10% টিন পোৱা গ'ল। পিতলত তাম আৰু জিংকৰ শতকৰা অংশ নিৰ্ণয় কৰা।

Ans : Let brass has $a$% copper and $b$% zinc. Let the mixed mass has $x$ quantity of brass and $y$ quantity of bronze. Comparing the amount of tin, we get $\frac{16}{100}y = \frac{10}{100}(x+y)$ i.e. $6y = 10x$. Comparing amount of copper, we get $\frac{a}{100}x + \frac{80}{100}y = \frac{74}{100}(x+y)$ i.e. $6y = (74-a)x$ i.e. $10x = (74-a)x$ which gives $a = 64$. Thus, $b = 36$. So, brass is 64% copper and 36% zinc.

14. If $x$ is a real number then solve $x^2(2-x)^2 = 1 + 2(1-x)^2$.

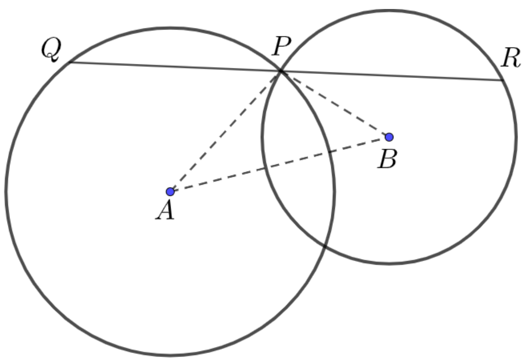
যদি $x$ এটা বাস্তৱ সংখ্যা, তেন্তে সমাধান কৰা: $x^2(2-x)^2 = 1 + 2(1-x)^2$।

Ans :

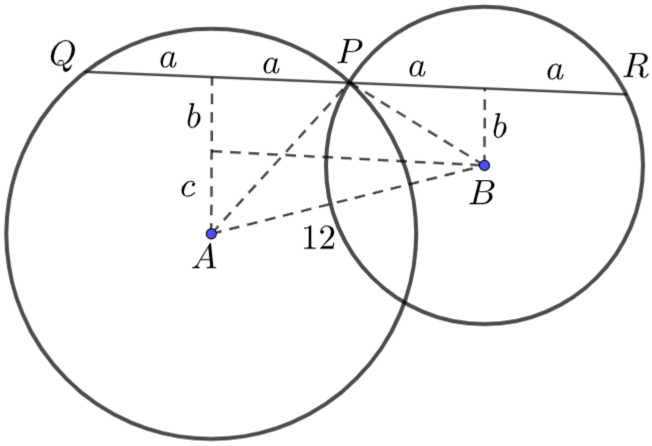
$$
\begin{aligned}
& x^2(2-x)^2 = 1 + 2(1-x)^2 \\
\Rightarrow & (2x - x^2)^2 = 1 + 2(1 - 2x + x^2) \\
\Rightarrow & (2x - x^2)^2 = 3 - 2(2x - x^2) \\
\Rightarrow & t^2 + 2t - 3 = 0 \text{ where } t = 2x - x^2 \\
\Rightarrow & (t+3)(t-1) = 0 \\
\Rightarrow & t = -3, t = 1 \\
\Rightarrow & 2x - x^2 = -3, 2x - x^2 = 1 \\
\Rightarrow & x^2 - 2x - 3 = 0, x^2 - 2x + 1 = 0 \\
\Rightarrow & x = 3, -1, x = 1, 1
\end{aligned}
$$

15. In the following figure two circles of radii $8$ and $6$ units are drawn with their centers $12$ units apart. At $P$, one of the points of intersections a line is drawn in such a way that the chords $QP$ and $PR$ have equal length. Find $QP^2$.

তলৰ চিত্ৰত $8$ আৰু $6$ একক ব্যাসাৰ্ধৰ দুটা বৃত্ত সিহঁতৰ কেন্দ্ৰ দুটাৰ দূৰত্ব $12$ একক হোৱাকৈ অংকন কৰা হৈছে। বৃত্ত দুটাৰ এটা ছেদবিন্দু $P$ ৰে এডাল ৰেখা এনেদৰে অঁকা হৈছে যাতে $QP$ আৰু $PR$ জ্যা দুডালৰ দৈৰ্ঘ্য সমান। $QP^2$ নিৰ্ণয় কৰা।




Ans : We know, perpendicular from the centre to the chord bisects the chord. Thus, $BP^2 = a^2 + b^2$ i.e. $a^2 + b^2 = 36$. Also, $AP^2 = a^2 + (b+c)^2$ i.e. $a^2 + (b+c)^2 = 8^2 = 64$. Again $AB^2 = (2a)^2 + c^2$ i.e. $4a^2 + c^2 = 144$. So, $4a^2 + c^2 - 4(a^2 + b^2) = 144 - 4 \times 36 = 0$. This gives $c = 2b$. So, $a^2 + (b + 2b)^2 = 64$ i.e. $a^+ 9b^2 = 64$ so that $(a^2 + b^2) + 8b^2 = 64$ i.e. $36 + 8b^2 = 64 \Rightarrow b^2 = 28/8 = 7/2$. So, $a^2 = 36 - 7/2 = 65/2$. Thus, $QP^= (2a)^2 = 4a^2 = 4 \times (65/2) = 130$.




16. Prove that if $x^2 + y = y^2 + z = z^2 + x$ then $x^3 + y^3 + z^3 = xy^2 + yz^2 + zx^2$, where $x$, $y$, $z$ are real numbers.

    যদি $x^2 + y = y^2 + z = z^2 + x$, তেন্তে প্ৰমাণ কৰা যে $x^3 + y^3 + z^3 = xy^2 + yz^2 + zx^2$ , য'ত $x$, $y$, $z$ বাস্তৱ সংখ্যা।

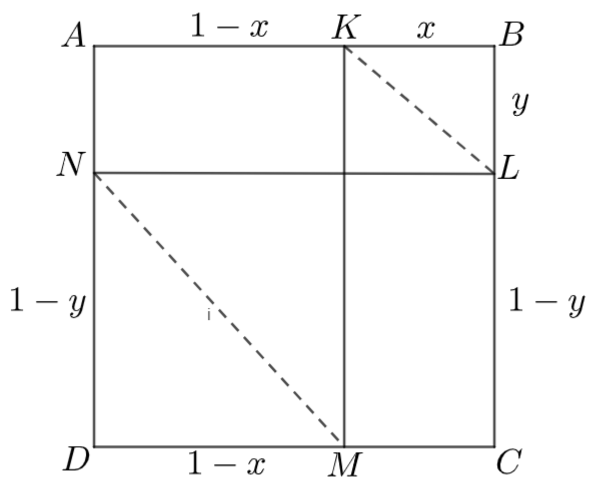
    Ans : Given $x^2 + y = y^2 + z = z^2 + x$. So, $x^2 - y^2 = z - y, y^2 - z^2 = x - z, z^2 - x^2 = y - x$.

$$\begin{aligned}(x^3 + y^3 + z^3) - (xy^2 + yz^2 + zx^2) =& x(x^2 - y^2) + y(y^2 - z^2) + z(z^2 - x^2)\\ =& x(z - y) + y(x - z) + z(y - x)\\ =& xz - xy + yx - yz + yz - zx = 0\end{aligned}$$

    Hence, $x^3 + y^3 + z^3 = xy^2 + yz^2 + zx^2$.

17. $K, L, M, N$ are points on sides $AB, BC, CD$ and $DA$ respectively of a unit square $ABCD$ such that $KM \parallel BC$ and $LN \parallel AB$. The perimeter of triangle $KLB$ is 1 unit. Find the area of triangle $MND$.

    $K, L, M, N$ হৈছে এটা একক কালিৰ বৰ্গ $ABCD$ ৰ ক্ৰমে $AB, BC, CD$ আৰু $DA$ বাহুৰ ওপৰৰ বিন্দু যাতে $KM \parallel BC$ আৰু $LN \parallel AB$। ত্ৰিভুজ $KLB$ ৰ পৰিধি 1 একক। ত্ৰিভুজ $MND$ ৰ ক্ষেত্ৰফল নিৰ্ণয় কৰা।



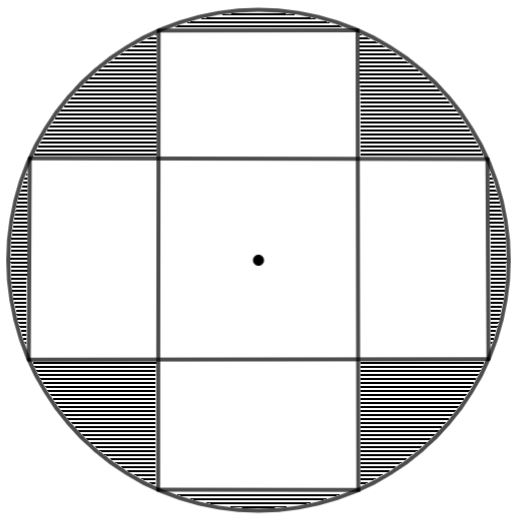
Ans : Since perimeter of $\Delta KLB$ is 1, so $KL+x+y=1$ i.e. $KL=1-x-y$. By Pythagoras theorem in $\Delta KLB$, we have

$$(1-x-y)^2 = x^2+y^2$$
$$\Rightarrow 1-2x-2y+2xy=0$$

$$\begin{aligned}\text{Area of } \Delta MND &= \frac{1}{2}(1-x)(1-y)\\ &= \frac{1}{2}(1-x-y+xy)\\ &= \frac{1}{4}(1+1-2x-2y+2xy)\\ &= \frac{1}{4}(1+0)\\ &= \frac{1}{4}\end{aligned}$$

18. Two $10 \times 24$ rectangles are inscribed in a circle as shown bellow. Let $A$ be the area of the shaded region. Assuming $\pi = \frac{22}{7}$, Find $7A$.

দুটা $10 \times 24$ আয়তক্ষেত্ৰ এটা বৃত্তত ভিতৰত তলত দেখুওৱাৰ দৰে ৰখা হৈছে। ধৰা, $A$ হৈছে ছাঁ দিয়া অংশৰ ক্ষেত্ৰফল। $\pi = \frac{22}{7}$ ধৰি লৈ, $7A$ নিৰ্ণয় কৰা।



Ans : Given $A$ is the area of the shaded region. Let $B$ be the area of the unshaded region. Then,

$$B = 10\times 24 + 10\times 24 - 10\times 10$$
$$\Rightarrow B = 380$$

If $r$ is the radius of the circle, then $(2r)^2 = 10^2+24^2 = 676$ so that $r^2 = 169$.

Now,

$$
\begin{aligned}
&A + B = \pi r^2 \\
\Rightarrow &A + 380 = \frac{22}{7} \times 169 \\
\Rightarrow &7A + 2660 = 3718 \\
\Rightarrow &7A = 1058
\end{aligned}
$$