Assam Academy of Mathematics
Assam Mathematics Olympiad 2022
Category III (Classes IX - XI)

Full marks : 100 Time : 3 hours

There are 18 questions. Questions 1 to 5 carry 2 marks each. Questions 6 to 13 carry 5 marks each. Questions 14 to 18 carry 10 marks each.

ইয়াত 18 টা প্ৰশ্ন আছে। 1 ৰ পৰা 5 লৈ প্ৰতিটো প্ৰশ্নত 2 নম্বৰকৈ আছে। 6 ৰ পৰা 13 লৈ প্ৰতিটো প্ৰশ্নত 5 নম্বৰকৈ আছে। আৰু 14 ৰ পৰা 18 লৈ প্ৰতিটো প্ৰশ্নত 10 নম্বৰকৈ আছে।

There may be various other ways of solutions than those shown here. Queries or suggestions regarding the solutions can be mailed to mail@aamonline.in

1. For the positive integers a, b, $\text{lcm}(a, b) = \text{gcd}(a, b) = p^2q^4$, where p and q are prime numbers. Find $\text{lcm}(ap, bq)$. Here lcm and gcd represent the least common multiple and the greatest common divisor respectively.

   দুটা ধনাত্মক পূৰ্ণসংখ্যা a, b ৰ কাৰণে $\text{lcm}(a, b) = \text{gcd}(a, b) = p^2q^4$, য'ত p আৰু q মৌলিক সংখ্যা। $\text{lcm}(ap, bq)$ নিৰ্ণয় কৰা। ইয়াত lcm আৰু gcd এ ক্ৰমে লঘিষ্ঠ সাধাৰণ গুণিতক আৰু গৰিষ্ঠ সাধাৰণ গুণনীয়ক বুজাইছে।

   Ans : Since $\text{lcm}(a, b) = \text{gcd}(a, b)$, so $a = b$. So, ap and bq have two more uncommon factors p and q. So, $\text{lcm}(ap, bq) = (p^2q^4)pq = p^3q^5$.

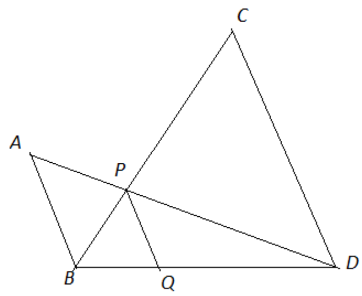
2. Find the sum of all the positive divisors of 27000.

   27000 ৰ আটাইকেইটা ধনাত্মক উৎপাদকৰ যোগফলটো নিৰ্ণয় কৰা।

   Ans : $27000 = 2^3 \times 3^3 \times 5^3$. So, sum of the divisors $\sigma(27000) = (1 + 2 + 2^2 + 2^3)(1 + 3 + 3^2 + 3^3)(1 + 5 + 5^2 + 5^3) = 93600$.

3. Let AB and CD be two parallel lines connected by the base BD. AD and BC are drawn and they intersect at the point P. A line PQ is drawn from the point P to BD such that $\angle PAB = \angle DPQ$. Prove that $\dfrac{1}{AB} + \dfrac{1}{CD} = \dfrac{1}{PQ}$.

   ধৰা, AB আৰু CD দুডাল সমান্তৰাল ৰেখা আৰু সিহঁতক BD ভূমিয়ে সংযোগ কৰিছে। AD আৰু BC অংকন কৰা হ'ল আৰু সিহঁতে P বিন্দুত কাটিছে। PQ ৰেখাডাল P ৰ পৰা BD লৈ অংকন কৰা হ'ল যাতে $\angle PAB = \angle DPQ$। প্ৰমাণ কৰা যে $\dfrac{1}{AB} + \dfrac{1}{CD} = \dfrac{1}{PQ}$।



Ans : $\angle PAB = \angle DPQ$ gives $PQ || AB$ and so $PQ || CD$. $\Delta DPQ \sim \Delta DAB$. So, $\frac{PQ}{AB} = \frac{DQ}{DB}$. Again, $\Delta BPQ \sim \Delta BCD$ gives $\frac{PQ}{CD} = \frac{BQ}{BD}$. Adding, we get $\frac{PQ}{AB} + \frac{PQ}{CD} = \frac{DQ}{BD} + \frac{BQ}{BD}$ i.e. $\frac{1}{AB} + \frac{1}{CD} = \frac{1}{PQ}$.

4. Evaluate

   নিৰ্ণয় কৰা

$$\frac{1}{1!21!} + \frac{1}{3!19!} + \frac{1}{5!17!} + \cdots + \frac{1}{21!1!}.$$

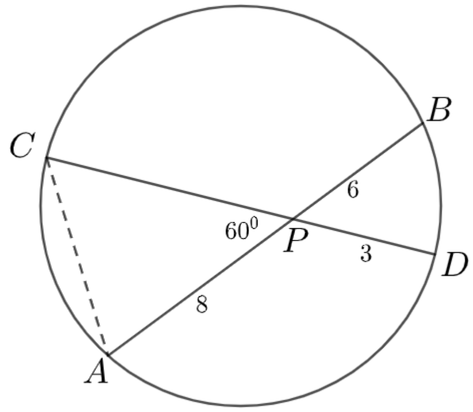
   Ans : The expression is evaluated as

$$\begin{aligned} & \frac{1}{22!}\left(\frac{22!}{1!21!} + \frac{22!}{3!19!} + \frac{22!}{5!17!} + \cdots + \frac{22!}{19!3!} + \frac{22!}{21!1!}\right) \\ =& \frac{1}{22!}\left(\binom{22}{1} + \binom{22}{3} + \cdots + \binom{22}{21}\right) \\ =& \frac{2^{22-1}}{22!} \text{ , since } \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \ldots = 2^{n-1} \\ =& \frac{2^{21}}{22!} \end{aligned}$$

   Hence, the original expression evaluates to $\frac{2^{21}}{22!}$

5. Chords AB and CD of a circle intersect inside the circle at P such that $AP = 8, PB = 6, PD = 3$ and $\angle APC = 60°$. Find the area of ΔAPC.

   এটা বৃত্তৰ দুডাল জ্যা AB আৰু CD এ বৃত্তটোৰ ভিতৰত P বিন্দুত এনেদৰে কাটিছে যাতে $AP = 8, PB = 6, PD = 3$ আৰু $\angle APC = 60°$। ΔAPC ৰ কালি নিৰ্ণয় কৰা।



Ans : By property of intersecting chords, we have $AP \times PB = CP \times PD$ so that $CP = \dfrac{8 \times 6}{3} = 16$. So, area of $\Delta APC = \dfrac{1}{2} AP \times CP \sin \angle APC = \dfrac{1}{2} \times 8 \times 16 \sin 60^0 = 4 \times 16 \times \dfrac{\sqrt{3}}{2} = 32\sqrt{3}$.

6. Prove that $n! \geq n^{n/2}$ for all natural numbers n. Also, show that the inequality is strict for $n > 2$. (In the question paper, it was typed $n \geq 2$ by mistake.)
প্ৰমাণ কৰা যে সকলো স্বাভাৱিক সংখ্যা n ৰ বাবে $n! \geq n^{n/2}$। আৰু দেখুওৱা যে $n \geq 2$ ৰ বাবে অসমতাটো কঠোৰ (strict)।

Ans :

$$\begin{aligned}(n!)^2 &= (1 \cdot 2 \cdot 3 \cdots n)(n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1) \\ &= \{1 \cdot n\}\{2 \cdot (n-1)\}\{3 \cdot (n-2)\} \cdots\cdots \{(n-1) \cdot 2\}\{n \cdot 1\}\end{aligned}$$

For $1 \leq r \leq n$, we have $r(n-r+1)-n = rn-r^2+r-n = n(r-1)-r(r-1) = (n-r)(r-1) \geq 0$. So, $r(n-r+1) \geq n$. The inequality is strict when $1 < r < n$. Thus,

$$\begin{aligned}(n!)^2 &= \{1 \cdot n\}\{2 \cdot (n-1)\}\{3 \cdot (n-2)\} \cdots\cdots \{(n-1) \cdot 2\}\{n \cdot 1\} \\ &\geq n \cdot n \cdot \cdots\cdots n \text{ (n times)} \\ &= n^n\end{aligned}$$

Thus, $n! \geq n^{n/2}$. For $n > 2$, the inequality is strict as $2(n-1) > n$.

7. In how many ways can 10 balls of same size be distributed among 4 children in the following cases:

   (a) all the balls are of the same colour?

   (b) each ball is of a different colour?

   একে আকৃতিৰ 10 টা বল 4 গৰাকী শিশুৰ মাজত কিমান ধৰণে ভগাই দিব পাৰিবা, যদি:

   (a) আটাইকেইটা বল একে ৰঙৰ হয়?

(b) প্ৰতিটো বল পৃথক পৃথক ৰঙৰ হয়?

Ans : If all the balls are of same colour then they can be considered identical. Number of ways of distributing 10 identical balls among 4 children is equal to the number non-negative integral solutions of $w+x+y+z=10$ which is equal to $\binom{10+4-1}{10}=\binom{13}{10}=\frac{13\times 12\times 11}{3\times 2\times 1}=286$. If the balls are all of different colours, then each ball can be distributed in 4 ways. So, total number of ways is $4^{10}=1048576$.

8. Let $a_1, a_2, a_3, \ldots, a_{2022}$ be positive real numbers which can be grouped into 1011 pairs such that each number of a pair is the reciprocal of the other number. Show that $(1+a_1)(1+a_2)(1+a_3)\cdots(1+a_{2022}) \geq 2^{2022}$.

ধৰা $a_1, a_2, a_3, \ldots, a_{2022}$ ধনাত্মক বাস্তৱ সংখ্যা যিসমূহক 1011 টা যোৰত থুপ কৰিব পৰা যাব যাতে একোটা যোৰৰ প্ৰতিটো সংখ্যা পৰস্পৰ বিপ্ৰতীপ। দেখুওৱা যে $(1+a_1)(1+a_2)(1+a_3)\cdots(1+a_{2022}) \geq 2^{2022}$।

Ans : Since each number of a pair is the reciprocal of the other number, the product of the numbers in each pair is 1. Hence, $a_1a_2\cdots a_{2022}=1$. AM-GM inequality gives $\frac{1+a_i}{2} \geq \sqrt{1\cdot a_i}$ i.e. $(1+a_i) \geq 2\sqrt{a_i}$ for each $i = 1, 2, \ldots, 2022$. Multiplying, we get $(1+a_1)(1+a_2)(1+a_3)\cdots(1+a_{2022}) \geq 2^{2022}\sqrt{a_1a_2\cdots a_{2022}} = 2^{2022}$.

9. What is the number formed by the last three digits of 1201^{1202}?

1201^{1202} ৰ শেষৰ অংক তিনিটাৰে গঠিত সংখ্যাটো কি?

Ans : Euler's theorem says if $gcd(a,n)=1$ then $a^{\phi(n)} \equiv 1 \pmod{n}$. The last three digits can be obtained from the remainder when the number is divided by 1000. Here, $gcd(1201, 1000) = 1$ so that $1201^{\phi(1000)} \equiv 1 \pmod{1000}$. Now, $\phi(1000) = \phi(2^3\times 5^3) = (2^3-2^2)(5^3-5^2) = 400$. So, $1201^{400} \equiv 1 \pmod{1000}$. Hence, $1201^{1202} \equiv (1201^{400})^3 \times 1201^2 \pmod{1000} \equiv 201^2 \pmod{1000} \equiv 401 \pmod{1000}$. Thus, the last three digits give the number 401.

10. Let the vertices of the square $ABCD$ are on a circle of radius r and with center O. Let P, Q, R and S are the mid points of AB, BC, CD and DA respectively. Then;

(a) Show that the quadrilateral $PQRS$ is a square.

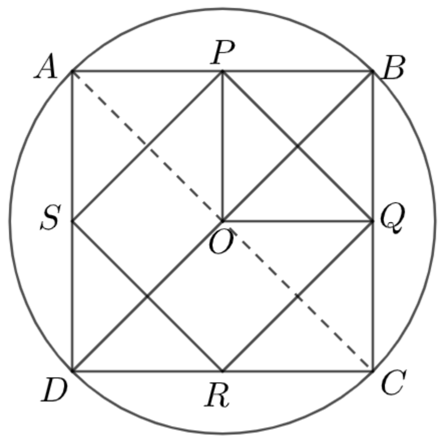
(b) Find the distance from the mid point of PQ to O.

ধৰা, $ABCD$ বৰ্গটোৰ শীৰ্ষবিন্দুকেইটা এটা বৃত্তত আছে যাৰ ব্যাসাৰ্ধ r আৰু কেন্দ্ৰ O। ধৰা, P, Q, R আৰু S ক্ৰমে AB, BC, CD আৰু DA ৰ মধ্যবিন্দু। তেন্তে;

(a) দেখুওৱা যে $PQRS$ চতুৰ্ভুজটো এটা বৰ্গ।

(b) PQ ৰ মধ্যবিন্দুৰ পৰা O লৈ দূৰত্ব নিৰ্ণয় কৰা।

Ans :



Note that $\triangle ASP$ is a isosceles triangle right angle triangle. Hence $\angle ASP = \angle APS = \pi/4$. Similarly $\angle BPQ = \angle BQP = \pi/4$. Now P is a point on the line AB. Hence:

$$\begin{aligned} &\angle APS + \angle SPQ + \angle BPQ = \pi \\ \Rightarrow &\pi/4 + \angle SPQ + \pi/4 = \pi/2 \\ \Rightarrow &\angle SPQ = \pi/2 \end{aligned}$$

Similarly we can show that $\angle PQR = \angle QRS = \angle RSP = \angle SPQ = \pi/2$. Hence $PQRS$ is a rectangle.

Since ABCD is a square in a circle, so its diagonals AC abd BD must be diameters. We know, line joining the mid-points of two sides of a triangle is half the third side. So, $PQ = RS = AC/2 = r$ and $QR = SP = BD/2 = r$. This gives $PQ = QR = RS = SP$. So, PQRS is a square.

O is the centre of the circle. Join OP, OQ and OB. The line joining the centre of a circle to the mid-point of a chord is perpendicular to the chord. Hence, $\angle OPB = \angle OQB = \pi/2$. Thus, ΔBPO and ΔBQO are isosceles right angled triangle. So, $OP = PB = BQ = OQ$. Thus, OPBQ is also a square. So, diagonals OB and PQ bisect each other at 90^0. So, the perpendicular distance from the mid point of PQ to O is $OB/2$ i.e. $r/2$.

11. Let a, b, c be the sides of a triangle such that $\frac{a^2+b^2+c^2}{ab+bc+ca}$ is an integer. Find the relation between a, b, c.

ধৰা, a, b, c এটা ত্ৰিভুজৰ বাহুসমূহৰ দৈৰ্ঘ্য যাতে $\frac{a^2+b^2+c^2}{ab+bc+ca}$ এটা অখণ্ড সংখ্যা। a, b, c ৰ মাজৰ সম্পৰ্ক নিৰ্ণয় কৰা।

Ans : Firstly we the have the obvious inequality $(a-b)^2+(b-c)^2+(c-a)^2 \geq 0$,

$$\begin{aligned} &\implies 2a^2+2b^2+2c^2-2ab-2bc-2ca \geq 0 \\ &\implies a^2+b^2+c^2-ab-bc-ac \geq 0 \\ &\implies \frac{a^2+b^2+c^2}{ab+bc+ca} \geq 1 \end{aligned}$$

Now, from properties of triangle we have the following inequalities,

$$|a-b| < c \tag{1}$$
$$|b-c| < a \tag{2}$$
$$|a-c| < b \tag{3}$$

Now, squaring and adding $(1),(2),(3)$ we have,

$$\begin{aligned} &\implies 2a^2+2b^2+2c^2-2ab-2bc-2ca < a^2+b^2+c^2 \\ &\implies a^2+b^2+c^2 < 2ab+2bc+2ca \\ &\implies \frac{a^2+b^2+c^2}{ab+bc+ca} < 2 \end{aligned}$$

Thus, the only integer $\frac{a^2+b^2+c^2}{ab+bc+ca}$ can attain is 1 which is possible only when the first inequality is an equality. Hence, $a=b=c$ and the triangle is an equilateral triangle.

12. A particle is in the origin of the Cartesian plane. In each step the particle can go 1 unit in any of the directions, left, right, up or down. Find the number of ways to go from $(0,0)$ to $(0,2)$ in 6 steps. (Note: Two paths where identical set of points is traversed are considered different if the order of traversal of each point is different in both paths.)

এটা কণা কাৰ্টেছিয়ান সমতলৰ কেন্দ্ৰত আছে। কণাটোৱে স্থান সলনি কৰোঁতে প্ৰতিবাৰে বাওঁ, সোঁ, ওপৰ বা তললৈ যিকোনো দিশত 1 একক যাব পাৰে। কণাটোৱে $(0,0)$ ৰ পৰা $(0,2)$ লৈ 6 টা ধাপত কিমান ধৰণে যাব পাৰিব নিৰ্ণয় কৰা। (বিঃদ্ৰঃ একে একে বিন্দু অতিক্ৰম কৰা দুটা পথক সুকীয়া বুলি ধৰা হ'ব যদি দুয়োটা পথতে প্ৰতিটো বিন্দু অতিক্ৰমৰ ক্ৰম সুকীয়া হয়।

Solution Denote R, U, L, D for right, up, left, down respectively. Hence, each path is a permutation of a sequence of R, U, L, D. Let r, u, l, d be the number of R, U, L, D respectively in a given sequence. Since the total number of steps is 6, so $r+l+u+d=6$. Since we are to reach $(0,2)$ from $(0,0)$, so there is no net horizontal (parallel to X-axis) movement. Thus, there should be equal number of right and left steps i.e. $r=l$. Also, the net vertical (along positive Y-axis) movement should be 2. So, $u-d=2$. Therefore, we have the following constraints on r, u, l, d

$$\begin{aligned} r + l + u + d &= 6 \\ r - l &= 0 \\ u - d - 2 &= 0 \end{aligned}$$

which simplifies to $r + d = 2$. Hence, we have the following 3 cases

Case 1: $r = 0, d = 2$. Then, $l = 0, u = 4$.

Here we have 4 U's, 2 D's, and no L, R's. Thus, the number of permutations for this case is $\frac{6!}{4!2!}$

Case 2: $r = 1, d = 1$. Then, $l = 1, u = 3$

Here we have 3 U's, 1 L, 1 R, 1 D. Total permutations= $\frac{6!}{3!}$

Case 3: $r = 2, d = 0$. Then, $l = 2, u = 2$.

Here we have 2 R's, 2 U's, 2 L's. Total permutations= $\frac{6!}{2!2!2!}$

Hence, the total number of paths is $\frac{6!}{4!2!} + \frac{6!}{3!} + \frac{6!}{2!2!2!} = 225$.

13. Calculate the given expression
প্ৰদত্ত ৰাশিটোৰ মান নিৰূপন কৰা

$$\sum_{k=0}^{n} \frac{2^k}{3^{2^k} + 1}$$

Solution We have,

$$\begin{aligned} &= \sum_{k=0}^{n} \frac{2^k(3^{2^k} - 1)}{3^{2^{k+1}} - 1} \\ &= \sum_{k=0}^{n} \frac{2^k(3^{2^k} + 1) - 2^{k+1}}{3^{2^{k+1}} - 1} \\ &= \sum_{k=0}^{n} \frac{2^k}{3^{2^k} - 1} - \frac{2^{k+1}}{3^{2^{k+1}} - 1} \\ &= \frac{2^0}{3^{2^0} - 1} - \frac{2^{n+1}}{3^{2^{n+1}} - 1} = 1 - \frac{2^{n+1}}{3^{2^{n+1}} - 1} \end{aligned}$$

14. The following sum of three four digits numbers is divisible by 75,

$$7a71 + 73b7 + c232,$$

where a, b, c are decimal digits. Find the necessary conditions in a, b, c.

চাৰিটা অংকৰে গঠিত তিনিটা সংখ্যা যোগ কৰা হৈছে:

$$7a71 + 73b7 + c232 ।$$

এই যোগফলটো 75 ৰে হৰণ যায়, য'ত a, b, c দশমিক অংক। ইয়াত a, b, c ৰ বাবে প্ৰয়োজন হোৱা চৰ্তসমূহ নিৰ্ণয় কৰা।

Ans : The prime factorization of 75 is $3 \times 5 \times 5$. We see that the unit's digit of the sum is 0. Therefore, the ten's digit of the sum must be 0 or 5, since the sum is divisible by 25. The ten's digit of the sum is given by $7 + b + 3 + 1 = 11 + b$. Now, we have two cases.

Case-I: The units digit of $11 + b$ is 0. So, $1 + b$ must be 10. Hence $b = 9$. Again, the sum is divisible by 3. Therefore $(7+a+7+1)+(7+3+b+7)+(c+2+3+2) = a + c + 48$ is divisible by 3. Therefore, $a + c$ is divisible by 3. So, in this case, the conditions are $c \neq 0, b = 9$ and $a + c$ is divisible by 3.

Case-II: The units digit of $11 + b$ is 5. So, $b = 4$. Again, the sum is divisible by 3. Therefore $(7 + a + 7 + 1) + (7 + 3 + b + 7) + (c + 2 + 3 + 2) = a + c + 43$ is divisible by 3. Therefore, $a + c + 1$ is divisible by 3. So, in this case, the conditions are $c \neq 0, b = 4$ and $a + c + 1$ is divisible by 3.

Hence, the required conditions are
$c \neq 0, b = 9$ and $a + c$ is divisible by 3,
or $c \neq 0, b = 4$ and $a + c + 1$ is divisible by 3.

15. Let $P(x)$ be a polynomial with positive integer coefficients and $P(0) > 1$. The product of all the coefficients is 47 and the remainder when $P(x)$ is divided by $(x - 3)$ is 9887. What is the degree of $P(x)$?

ধৰা, $P(x)$ ধনাত্মক অখণ্ড সংখ্যা সহগ যুক্ত এটা বহুপদ ৰাশি আৰু $P(0) > 1$। ইয়াৰ সহগসমূহৰ পূৰণফল 47 আৰু $P(x)$ ক $(x - 3)$ ৰে হৰণ কৰিলে 9887 বাকী ৰয়। $P(x)$ ৰ মাত্ৰা কিমান?

Ans : Let $P(x) = a_0 + a_1x + a_2x^2 + \ldots\ldots + a_nx^n$. Since $a_0a_1a_2 \cdots a_n = 47$ and 47 is a prime, so all the coefficients are 1 except one of them which is 47. Given $P(0) > 1$ i.e. $a_0 > 1$. Hence, $a_0 = 47$. Also, $P(3) = 9887$, so $47 + 3 + 3^2 + 3^3 + \ldots\ldots + 3^n = 9887$ i.e. $\dfrac{3(3^n - 1)}{3 - 1} = 9840$ which gives $3^n - 1 = 6560$ i.e. $3^n = 6561 = 3^8$ i.e. $n = 8$.

16. Can we find a subset A of $\mathbb{N}$ containing exactly five numbers such that sum of any three elements of A is a prime number? Justify your answer.

স্বাভাৱিক সংখ্যাৰ সংহতি $\mathbb{N}$ ৰ পাঁচটা মৌল যুক্ত এটা উপ-সংহতি A পোৱা যাব নে যাতে A ৰ যিকোনো তিনিটা মৌলৰ যোগফল মৌলিক হয়? তোমাৰ উত্তৰটো সত্যাপন কৰা।

Ans : We know any natural number n is one of the following forms : $n = 3k$ or $n = 3k + 1$ or $n = 3k + 2$ for some natural number k. A has 5 elements and there are three possibilities:

(a) If all elements of A are of same type then the sum of any three elements will be greater than 3 and divisible by 3.

(b) If the 5 elements of A are of two distinct types, then from Pigeonhole principle at least three elements of A will be of the same type so that their sum is greater than 3 and divisible by 3.

(c) Suppose A has at least one element of each type. Then let $n_1 = 3k$, $n_2 = 3p + 1$ and $n_3 = 3q + 2$. Then $n_1 + n_2 + n_3 = 3(k + p + q) + 3$ which is greater than 3 and divisible by 3.

Hence we can't find a set $A \subseteq \mathbb{N}$ such that the sum of any three elements of A is a prime number because we have shown that any such set A will contain three elements whose sum is greater than 3 and divisible by 3.

17. Consider a rectangular grid of points consisting of 4 rows and 84 columns. Each point is coloured with one of the colours red, blue or green. Show that no matter whatever way the colouring is done, there always exist four points of the same colour that form the vertices of a rectangle. An illustration is shown in the figure below.

বিন্দুৰে গঠিত আয়তাকাৰ জালিকা এখনত 4 টা শাৰী আৰু 84 টা স্তম্ভ আছে। প্ৰতিটো বিন্দুক ৰঙা, নীলা বা সেউজীয়া কোনো এটা ৰঙেৰে ৰং কৰা হৈছে। দেখুওৱা যে, বিন্দুসমূহক যি ধৰণেই ৰং কৰা নহওক কিয়, সদায় একে ৰঙৰ চাৰিটা বিন্দু পোৱা যাব যি চাৰিটাই এটা আয়ত গঠন কৰে। তলৰ চিত্ৰত এটা ৰূপৰেখা দেখুওৱা হৈছে।

Ans : Each point in a column can be coloured in 3 ways. So, each column can be coloured in $3^4 = 81$ ways. Thus there are 81 possible colourings of the columns. But there are 84 columns and $84 > 81$. So, by pigeonhole principle, there exist two columns of the same colour pattern. We focus on these two columns of same colour pattern. Since there are four points in each column that are coloured with 3 colours, so by pigeonhole principle, two points in each column will be of same colour. Since the two columns have identical colour pattern, so there are four points of same colour that form the vertices of a rectangle.

18. Let $f : \mathbb{N} \to \mathbb{N}$ be a function such that

(a) $f(m) < f(n)$ whenever $m < n$.

(b) $f(2n) = f(n) + n$ for all $n \in \mathbb{N}$.

(c) n is prime whenever $f(n)$ is prime.

Find $\sum_{n=1}^{2022} f(n)$.

ধৰা $f : \mathbb{N} \to \mathbb{N}$ এটা ফলন যাতে

(a) $f(m) < f(n)$ যেতিয়া নেকি $m < n$।

(b) $f(2n) = f(n) + n$, $\forall\ n \in \mathbb{N}$।

(c) n টো মৌলিক যেতিয়া নেকি $f(n)$ মৌলিক।

$\sum_{n=1}^{2022} f(n)$ নিৰ্ণয় কৰা।

Ans : Observe that $f(2) = f(1) + 1$, $f(4) = f(2) + 2 = f(1) + 1 + 2 = f(1) + 3$. Now, by the first condition, $f(2) < f(3) < f(4)$. So, we have $f(1) + 1 < f(3) < f(1) + 3$. But $f(1) + 1$ and $f(1) + 3$ are natural numbers. So, we have $f(3) = f(1) + 2$ i.e. the natural number between $f(1) + 1$ and $f(1) + 3$. Let us assume the induction hypothesis that $f(r) = f(1) + r - 1$ for all $r < n$. If n is even then $n = 2k$ where $k < n$. Thus, $f(n) = f(2k) = f(k) + k = f(1) + k - 1 + k = f(1) + 2k - 1 = f(1) + n - 1$. If n is odd then $n = 2k + 1$ where $k < n$. Then, we have $f(2k) < f(2k + 1) < f(2k + 2)$ i.e. $f(k) + k < f(2k + 1) < f(k + 1) + k + 1$. But $k < n$ and $k + 1 < n$. Hence, we have $f(1) + k - 1 + k < f(2k + 1) < f(1) + k + k + 1$ which gives $f(1) + 2k - 1 < f(2k + 1) < f(1) + 2k + 1$. Hence we can conclude $f(n) = f(2k+1) = f(1)+2k = f(1)+n-1$. Thus, by Principle of Mathematical Induction, we have $f(n) = f(1) + n - 1$ for all $n \in \mathbb{N}$.

Now let $f(1) = m > 1$. Then, $m! + 2, m! + 3, \ldots, m! + m$ are all composite numbers. Let p be the smallest prime exceeding $m! + m$. Let $n = p - m + 1$. Then, $f(n) = f(1) + n - 1 = m + n - 1 = p$ which is prime. So, by the third condition, n is prime. Since $p = n + m - 1$ and $m > 1$, so $p > n$. Also, $n = p-m+1 > m!+m-m+1 = m!+1$. But $m!+2, m!+3, \ldots m!+m$ are all composite. So, $n > m! + m$. Thus we have $m! + m < n < p$. This contradicts the minimality of p. Thus$m > 1$ is not possible. So, $m = f(1) = 1$. Hence, $f(n) = f(1) + n - 1 = 1 + n - 1 = n$. Thus,

$$\begin{aligned}\sum_{n=1}^{2022} f(n) &= \sum_{n=1}^{2022} \\ &= \frac{2022(2022+1)}{2} \\ &= 2045253\end{aligned}$$