Assam Academy of Mathematics

MATHEMATICS OLYMPIAD

26th May, 2019

Category-III : (For Classes- IX, X & X appeared)

The figures in the margin indicate full marks for the questions

Total Marks : 100 Time : 10.00 AM – 1.00 PM

Answer all the questions:

1. Find the 5000th term of the following sequence: 5

1,2,2,3,3,3,4,4,4,4,5,5,5,5,5,...

Soln.

We observe that 1 appears once, 2 appears twice, 3 appears thrice, ..., n appears n times. Further, observe that 1st term is1, (1+2)th term is 2, (1+2+3)th term is 3.

So, (1+2+3+.... + n)th term is n.

Thus, $\frac{n(n+1)}{2}$th term is n.

Similarly, [1+2+3+ .... + (n-1)] th terms is

n-1 i.e. $\frac{n(n-1)}{2}$th term is $n-1$..

So, all terms from $\left(\frac{n(n-1)}{2}+1\right)$th term to $\frac{n(n+1)}{2}$th term are equal to n.

We try to find the value of n for which

$$\frac{n(n-1)}{2} < 5000 \leq \frac{n(n+1)}{2}$$

We see that 5000=50×100

$$= \frac{100\times100}{2} < \frac{100\times101}{2}$$

Also,

$$\frac{100\times99}{2} < \frac{100\times100}{2} = 5000 < \frac{100\times101}{2}$$

Thus, all terms from (50×99+1)th term to (50×101)the term are equal to 100.

∴The 5000th term is 100.

2. Let S = {(x,y,z) : 0#x, y, z # 9 and x+y+z is divisible by 3}. Find the number of elements of the set S. 6

Soln.

Any number is either of the form 3k, or 3k+1 or 3k + 2. i.e. any numbers leaves remainder 0 or 1 or 2 when divided by 3.

The numbers from 0 to 9 can be grouped into three categories acordingly. A={0,3,6,9}, B={1,4,7}, C={2,5,8}

If x,y,z ∈A, then x+y+z is divisible by 3.

No. of choices of (x,y,z) in that case is 4×4×4 (Multiplication rule)

If x,y,z ∈B, then x+y+z is divisible by 3.

∴ No. of choices = 3×3×3.

If x,y,z ∈C then x+y+z is divisible by 3

∴ No. of choices of (x,y,z) is 3×3×3.

The only other cases where x+y+z is divisible by 3 are those where each of x,y,z belong to different sets A,B,C. There are $\angle 3 = 6$ such cases. In each case, the no. of choices for (x,y,z) is 4×3×3.

∴The total no. of elements of the set S.

= 4×4×4+3×3×3+3×3×3+6×4×3×3

= 64+27+27+216

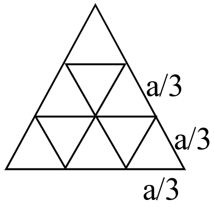
= 64+54+216

= 280+54

= 334

3. Show that out of any ten points chosen inside an equilateral triangle of side length a, there always exist two points whose distance apart is less than $\frac{a}{3}$ 6

Soln.



We trisect each side of the given equilateral triangle and join the points two at a time by line segments parallel to the side not containing the points. Thus, the whole area is divided into 9 smaller equilateral tringles each of side length a/3. Thus, marking ten points inside the triangle is equivalent to putting 10 objects in 9 boxes. Thus, one of the loopes will have two objects. This follows from pigeonhole principle. So, two points will be inside the same smaller triangle. Thus, their distance apart is less than a/3.

4. If *f* :ù6ù is a function satisfying *f(f(n)) + f (n+1)=n+2* œ*n*0ù, find the values of *f*(1) and *f*(2). 7

soln.

Taking n = 1,

f(*f*(1)) + *f*(2)=3

Since the codomain of *f* is ù, so the only possibilities are:

Case I: *f*{*f*(1)}=1, *f*(2)=2

Case II : *f*{*f*(1)}=2, *f*(2)=1

We have,

f{*f*(n)}+*f*(n+1)=n+2 –(*)

$\Rightarrow f(n+1) = n+2-f\{f(n)\} \forall n \in \mathbb{N}$

$\Rightarrow f(n+1) \le n+2-1 \quad \{\because f(f(n)) \ge 1\}$

$\Rightarrow f(n+1) \le n+1 \quad \forall n \in \mathbb{N}$

Similarly,

f(f(n)) ≤ n+1 œ*n*0ù ________ (2)

By (1), f(n) ≤ n œ*n* ≥ *2* ________ *(3)*

So for n ≥ 2 and for f(n) ≥ 2,

f(*f*(n)) ≤ *f*(n) ≤ n

⇒ *f*(*f*(n))-n ≤ O

⇒ 2-*f*(n+1) ≤ O

⇒ *f*(n+1) ≥ 2

Thus, if n ≥ 2 and f(n)) ≥ 2

then *f*(n+1) ≥ 2.

Consider case I : *f*(*f*(1))=1, *f*(2)=2

By above,

f(n) ≥ 2 œ n ≥ 2

Let *f*(1) = c

⇒ *f*(*f*(1)) = *f*(c)

⇒ 1= *f*(c)

⇒ c < 2 (∵ *f*(c) ≥ 2 if c ≥ 2)

⇒ c = 1

∴ *f*(1) = 1.

Consider case II:

f(*f*(1)) = 2, *f*(2) = 1

Let *f*(1) = c

⇒ *f*(*f*(1)) = *f*(c)

⇒ 2 = *f*(c)

Putting n = 2 in (*)

$f(f(2)) + f(3) = 4$

$\Rightarrow f(1) + f(3) = 4$

$\Rightarrow f(3) = 4 - c$

$\because f$: ù => ù, so $f(3) \geq 1$

$\Rightarrow 4 - c \geq 1$

$\Rightarrow c \leq 3$

If c = 1, then $f(1)$ = C and 2 = $f(c)$ gives $f(1) = 1$ and 2 = $f(1)$ which is not possible.

If c = 2, then 2 = f (c) gives $f(2) = 2$ but it contradicts $f(2) = 1$.

If c = 3, then 2 = $f(c)$ gives

$f(3) = 2$ but $f(3) = 4 - c$

gives $f(3) = 4 - 3 = 1$.

Thus none of these are possible.

Hence, $f(1) = 1$, $f(2) = 2$.

5. Let $f(x)=x^3+ax^2+bx+c$ and $g(x)=x^3+bx^2+cx+a$, where a b c are integers with c...0. Suppose that the following conditions are satisfied:

(a) $f(1)=0$

(b) the roots of $g(x)=0$ are squares of the roots of $f(x)=0$.

Find the value of $a^{2019}+b^{2019}+c^{2019}$. 7

Soln.

$f(1)-0$

$\Rightarrow 1+a+b+c=0$

$\Rightarrow a+b+c=-1$

Let roots of $f(x)=0$ be α,β,γ

$\therefore$Roots of $g(x)=0$ are α^2, β^2 γ^2

$f(x)=x^3+ax^2+bx+c=0$

$\Rightarrow$ Sum of roots =-a

$\Rightarrow \alpha + \beta + \gamma = -a$

Sum of products of roots taken 2 at a time = b

$\Rightarrow \alpha\beta + \beta\gamma + \gamma\alpha = b$

Product of roots is -c

$\therefore \alpha\beta\gamma = - c$

Also,

$g(x) \equiv x^3 + bx^2 + cx + a = 0$

$\Rightarrow \alpha^2 + \beta^2 + \gamma^2 = - b$

$\Rightarrow (\alpha + \beta + \gamma)^2 -1 (\alpha\beta + \beta\gamma + \gamma\alpha) = - b$

$\Rightarrow a^2 - 2b = - b$

$\Rightarrow a^2 = b$

And

$\alpha^2\beta^2\gamma^2 = - a$

$\Rightarrow (\alpha\beta\gamma)^2 = - a$

$\Rightarrow (-c)^2 = - a$

$\Rightarrow c^2 = -a$

$\therefore a+b+c = -1$

$\Rightarrow -c^2 + a^2 + c = -1 \Rightarrow -c^2 + c^4 + c = - 1$

$\Rightarrow c^4 - c^2 + c + 1 = 0$

$\Rightarrow c^2(c^2-1) + (c+1) = 0$

$\Rightarrow c^2(c-1) (c+1) + (c+1) = 0$

$\Rightarrow (c+1) [c^2(c-1) +1] = 0$

$\Rightarrow (c+1) [c^3-c^2+1] = 0$

$\Rightarrow c = -1$ or $c^3 = c^2 - 1$.

But $c^3=c^2-1$. doesn't have integer solutions. If c is odd then c^2-1 is even. So c^3 is even but that is not correct as c is odd. If c is even, then c^2-1 is odd, so that c^3 is odd but that is not true as c is even.

Thus, $c = -1$

$\therefore\ a = -c^2 = -1$

$b = a^2 = (-1)^2 = 1$

$\therefore\ a^{2019} + b^{2019} + c^{2019}$

$= -1 + 1 - 1$

$= 1$

6. A positive integer has unit digit 6. If we erase this unit digit and place it in front of the remaining digits, we get 4 times the original number. Determine the smallest such positive integer. 7

Soln.

Let the integer be $10a+6$ (n digits)

$10^{n-1} \times 6 + a = 4\times(10a+6)$

$\Rightarrow\ (10^{n-1}-4) \times 6 = 39a$

$\Rightarrow\ (10^{n-1}-4) \times 2 = 13a$

$\therefore\ 13 \mid (10^{n-1}-4)$

So we need to find the smallest n for which $13 \mid 10^{n-1}-4$

i.e. $10^{n-1} \equiv 4 \bmod 13$

We have $10 \equiv -3 \bmod 13$

$\Rightarrow\ 10^2 = 9 \bmod 13$

$10^2 \equiv 9 \bmod 13$

$\Rightarrow (10^2)^2 \equiv 9^2 \bmod 13$

$\Rightarrow 10^4 \equiv 81 \bmod 13$

$\Rightarrow 10^4 \equiv 3 \bmod 13$

$\Rightarrow 10^5 \equiv 30 \bmod 13$

$\Rightarrow 10^5 \equiv 4 \bmod 13$

$\Rightarrow 10^{6-1} \equiv 4 \bmod 13$

$\therefore$ Least such n is $n=6$.

$\therefore\ 13a = 2 \times (10^{6-1}-4)$

$\Rightarrow 13a = 2 \times 99996$

$$\Rightarrow a = 2\times\frac{99996}{13}$$

$\Rightarrow a = 2 \times 7692$

$\Rightarrow\ a = 15384$

$\therefore$ The number is 153846

7. Write the number of perfect squares, perfect cubes and perfect fourth powers from 1 to 10^6 (both inclusive). How many of the numbers from 1 to 10^6 are neither perfect squares, nor perfect cubes nor perfect fourth powers? 3+5=8

Soln.

The perfect squares from 1 to 10^6 are $1^2, 2^2, 3^2, 4^2, \ldots, (10^3)^2$

So, there are 1000 perfect squares. The perfect cubes from 1 to 10^6 are $1^3, 2^3, 3^3, \ldots., (10^2)^3$

So, there are 100 perfect cubes. Clearly, the no. of perfect fourth powers will be less than 100.

$10^6 = 1000 \times 1000$

Perfect square nearest to 1000 is $31^2=961$.

$\therefore$ The fourth power nearest to 10^6

is $31^4 = 31^2\times31^2$

$= 961\times961$

$\therefore$ No. of perfect fourth powers upto 10^6 is 31.

(One mark for the 1st two and 2 marks for the fourth power)

Let A=Set of perfect squres from 1 to 10^6

B=Set of perfect cubes from 1 to 10^6

C=Set of perfect fourth powers from 1 to 10^6.

$\therefore$ n(A)=1000, n(B) = 100, n(c)=31.

$A \cap B$ is the set of perfect squares which are also perfect cubes & vice versa.

This set contains elements which are cubes of perfect squares or equivalently squares of perfect cubes.

$\therefore$ The elements in $A \cap B$ are $(1^2)^3, (2^2)^3, (3^2)^3, ..., (10^2)^3$. Thus, n($A \cap B$) = 10.

The elements in $B \cap C$ are

$(1^3)^4, (2^3)^4$ & $(3^3)^4$ (upto $30)^4$

n($B \cap C$)=3.

The elements in $A \cap C$ are just the perfect fourth powers as every perfect fourth power is also a perfect square.

$\therefore$ n($A \cap C$) = 31

The numbers which are perfect squares as well as perfect cubes as well as perfect fourth powers are

$1^{12}, 2^{12}$ and 3^{12} (i.e. power should be LCM of 2,3,4

$\therefore n(A \cap B \cap C) = 3$

$\therefore$ By inclusion-exclusion principle,

$n(A^c \cap B^c \cap C^c)$

$= n(S) - \{n(A) + n(B) + n(c)\} + \{n(A \cap B) + n(B \cap C) + n(A \cap C)\} - n(A \cap B \cap C)$

$= 10^6 - (1000+100+31)+(10+3+31)-3$

= 1000000-1100+10

= 998900+10

= 998910

8. Let *a,b,c,* be the lengths of the sides *BC,CA* and *AB* of a triangle *ABC.* Consider all the possibilities:

(a) ABC is acute angled triangle

(b) A is an acute angle in a right angled triangle

(c) A is an acute angle in an obtuse angled triangle

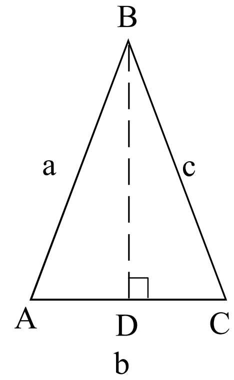
(d) A is an obtuse angle

(e) A is a right angle

In each case, prove that $a^2=b^2+c^2-2bc \cos A$. 2+1+2+2+1=8

Soln.

(a) ABC is acute angled $\triangle$



$BC^2=BD^2+DC^2$

$\Rightarrow a^2=AB^2-AD^2+DC^2$

$\Rightarrow a^2=c^2-AD^2+(AC-AD)^2$

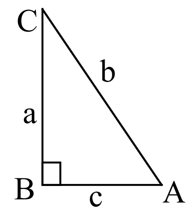
$\Rightarrow a^2=c^2-AD^2+AC^2-2AC.AD+AD^2$

$\Rightarrow a^2=c^2+AC^2-2.AC.AB \cos A$

$\Rightarrow a^2=c^2+b^2-2bc \cos A.$

(b) A is acute angle in a right angled $\triangle$

By Pythagoras theorm,



$a^2=b^2-c^2$

$\Rightarrow a^2=b^2+c^2-2c^2$

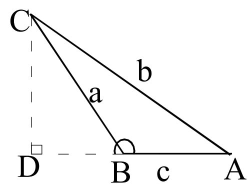
$\Rightarrow a^2=b^2+c^2-2.c.c$

$\Rightarrow a^2=b^2+c^2-2.(b \cos A)\, c.$

$\Rightarrow a^2=b^2+c^2-2bc \cos A.$

(c) A is an acute angle in an obtuse angled triangle.

In $\triangle$ BCD,



$a^2=CD^2+BD^2$

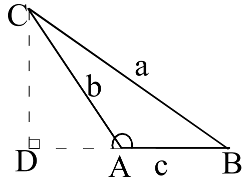
$\Rightarrow a^2=AC^2-AD^2+(AD-AB)^2$

$\Rightarrow a^2=b^2-AD^2+AD^2-2AD.AB+AB^2$

$\Rightarrow a^2=b^2+c^2-2.(AC \cos A).AB$

$\Rightarrow a^2=b^2+c^2-2bc \cos A.$

(d) A is an obtuse angle



In $\triangle BCD$,

$a^2=CD^2+BD^2$

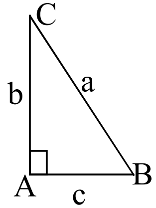
$\Rightarrow a^2=AC^2-AD^2+(BA+AD)^2$

$\Rightarrow a^2=b^2-AD^2+AB^2+AD^2+2AB.AD$

$\Rightarrow a^2=b^2+c^2+2.c.AC \cos (180-A)$

$\Rightarrow a^2=b^2+c^2-2bc \cos A.$ ($\therefore \cos (180-A)=-\cos A$)

(e) A is a right angle.



By Pythogoras theorem

$a^2=b^2+c^2$

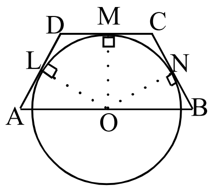
$\Rightarrow a^2=b^2+c^2-2bc.\cos 90$

$\Rightarrow a^2=b^2+c^2-2bc \cos A.$

9. A circle has centre on the side AB of a cyclic quadrilateral ABCD. The other three sides are tangents to the circle. Draw the diagram and prove that AD+BC=AB. 2+6=8

Soln.

(2 marks for diagram)



Since ABCD is cyclic.

So,

$\angle A+\angle C=180^0$

$\angle B+\angle D=180^0$

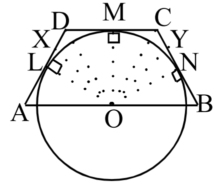
Let OL, OM and ON be the radii at the points of contact of the sides AD, DC and CB respectively. Then OL $\perp$ AD, OM $\perp$ DC, ON $\perp$ CB

Const : X and Y are marked on AD & BC such that

AX=AO and

BY=BO.

$\therefore \angle AXO = \left(\dfrac{180^0 - A}{2}\right) = 90^0 - \dfrac{A}{2}$



$\& \angle BYO = 900 - \dfrac{B}{2}$

In $\triangle OLX$ and $\triangle OCM$

$\angle LXO=\angle MCO=90^0-\dfrac{A}{2}$

$\angle OLX=\angle OMC$ (Each 90^0)

OL=OM(radii)

$\triangle OLX \cong \triangle OCM$

(AAS congruency)

$\therefore$ LX=MC

But MC=CN (tangents from C)

LX=CN

Similarly,

NY=DL

$\therefore$ AB=AO+OB

=AX + BY

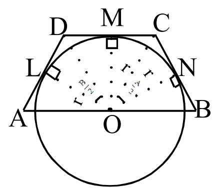
=AL+LX+BN+NY

=AL+CN+BN+DL

=(AL+DL)+(CN+BN)

=AD+BC

Another proof using trigonometry



$\angle LOM = 180^0 - \angle D$

$= \angle B$

$$\therefore \angle LOD = \angle DOM = \frac{\angle B}{2}$$

$$\angle NOC = \angle COM = \frac{\angle A}{2}$$

AD+BC=AL+LD+BN+NC

$$= r\left[\tan(90^0 - A) + \tan\frac{B}{2} + \tan(90^0 - B) + \tan\frac{A}{2}\right]$$

$$= r\left[\cot A + \tan\frac{B}{2} + \cot B + \tan\frac{A}{2}\right]$$

$$= r\left[\frac{1}{\tan A} + \tan\frac{A}{2} + \tan\frac{B}{2} + \frac{1}{\tan B}\right]$$

$$= r\left[\frac{1-\tan^2\frac{A}{2}}{2\tan\frac{A}{2}} + \tan\frac{A}{2} + \tan\frac{B}{2} + \frac{1-\tan^2\frac{B}{2}}{2\tan\frac{B}{2}}\right]$$

$$= r\left[\frac{1+\tan^2\frac{A}{2}}{2\tan\frac{A}{2}} + \frac{1+\tan^2\frac{B}{2}}{2\tan\frac{B}{2}}\right]$$

$$= r\left[\frac{1}{\sin A} + \frac{1}{\sin B}\right]$$

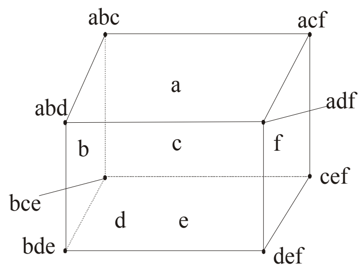
= r[CosecA + cosecB]

= AO + OB

= AB

10. Positive integers *a,b,c,d,e,f* are written on the six faces of a cube, one on each. At each of the eight corners (vertices), the product of the numbers on the faces that meet at that corner is written. The sum of the numbers written on the corners is 4444. Find all possible values of the sum of the numbers written on the faces. 8

Soln.



Sum of numbers written on the corners = abc+abd+acf+adf+bce+bde+def+cef

= ab(c+d)+af(c+d)+be(c+d)+cf(c+d)

= (c+d) [a(b+f)+e(b+f)]

= (c+d) (a+c) (b+f)

= 4444

= 4×1111

= 4×11×101

Since the intigers are positive, so each of the sums c+d, a+e and b+f must be greater than 1.

So, there are the following possibilities:

(c+d) (a+e) (b+f) = 4×11×101

So, (c+d)+(a+e)+(b+f) = 4+11+101

= 116

(c+d) (a+e) (b+f) = 2×22×+101

∴ (c+d)+(a+e)+(b+f) = 2+22+101

$= 125$

$(c+d)(a+e)(b+f)=2\times 11\times 202$

$\therefore (c+d)+(a+e)+(b+f)=2+11+202$

$= 215$

$\therefore (c+d)+(a+e)+(b+f)=2\times 2\times 1111$

$\therefore (c+d)+(a+e)+(b+f) = 2+2+1111$

$= 1115$

There are four possible values

11. Let x,y,z be non-negative real numbers such that $x+y+z=1$, prove that

$$0 \# xy+yz+zx-2xyz \# \frac{7}{27}.$$

(A hint : Put $x=a+\frac{1}{3}$, $y=b+\frac{1}{3}$, $z=c+\frac{1}{3}$ with appropriate restrictions on a, b and c.) 10

Soln.

Let $x=a+\frac{1}{3}, y=b+\frac{1}{3}, z=c+\frac{1}{3}$

$\therefore$ x+y+z=1

=> a+b+c=0

Since x,y,z are non-negative,

So $a \geq -\frac{1}{3}, b \geq -\frac{1}{3}, c \geq -\frac{1}{3}$

xy+yz+zx-2xyz

$$=\left(a+\frac{1}{3}\right)\left(b+\frac{1}{3}\right)+\left(b+\frac{1}{3}\right)\left(c+\frac{1}{3}\right)+\left(c+\frac{1}{3}\right)\left(a+\frac{1}{3}\right)-2\left(a+\frac{1}{3}\right)\left(b+\frac{1}{3}\right)\left(c+\frac{1}{3}\right)$$

$$\Rightarrow xy+yz+zx-2xyz$$

$$=ab+\frac{a+b}{3}+\frac{1}{9}+bc+\frac{b+c}{3}+\frac{1}{9}+ca+\frac{a+c}{3}+\frac{1}{9}-2\left(a+\frac{1}{3}\right)\left(bc+\frac{b+c}{3}+\frac{1}{9}\right)$$

$$=(ab+bc+ca)+\frac{2(a+b+c)}{3}+\frac{3}{9}$$

$$-2\left[abc+\frac{ab+ac}{3}+\frac{a}{9}+\frac{bc}{3}+\frac{b+c}{9}+\frac{1}{27}\right]$$

$$=(ab+bc+ca)+\frac{2}{3}\times 0+\frac{1}{3}-2abc-\left(\frac{ab+bc+ca}{3}\right)-\frac{1}{27}$$

$$=\frac{2}{3}(ab+bc+ca)-2abc+\frac{7}{27}$$

$$=\frac{2}{3}\left[ab+bc+ca-3abc\right]+\frac{7}{27}$$

$$=\frac{2}{3}\left[-\frac{1}{2}(a^2+b^2+c^2)-(a^3+b^3+c^3)\right]+\frac{7}{27}$$

$(\because a+b+c=0)$

$$=-\frac{1}{3}\left[a^2+b^2+c^2+2a^3+2b^3+2c^3\right]+\frac{7}{27}\leq\frac{7}{27}$$

as 1+2a, 1+2b, 1+2c $\geq$0

$(\because a,b,c \geq -\frac{1}{3})$

For the other part, we have xy+yz+zx-2xyz

=xy(1-z)+yz(1-x)+zx

=xy(x+y)+yz(y+z)+zx$\geq$0

12. By trial & error or otherwise, find four different solutions (a, b, c, n) in positive integers of the equation $2^n = a! + b! + c!$. Justify that these are the only possible solutions.

4+6=10

Soln.

One solution is a=1, b=1, c=2

$\therefore \angle a + \angle b + \angle c = 1 + 1 + 2 = 4 = 2^2$

i.e $n = 2$.

Observe that of a,b,c ≥ 3, then there exist no solutions as in that case, $\lfloor 3\left(\frac{\lfloor a}{\lfloor 3}+\frac{\lfloor b}{\lfloor 3}+\frac{\lfloor c}{\lfloor 3}\right)-2^n$

i.e. $3|2^n$ but that is not possible.

So, all three of a,b,c, cannot be greater than or equal to 3.

Again if a=b=c then $3|2^n$ which is not possible. So, a,b,c, cannot be all equal. without loss of generality,

Let $a \leq b \leq c$. Then

$$2^n = \lfloor a\left(1+\frac{\lfloor b}{\lfloor a}+\frac{\lfloor c}{\lfloor a}\right)$$

$\therefore \lfloor a \mid 2^n$

$\Rightarrow a = 1 \text{ or } 2$

If $a = 1$, then $2^n - 1 = \lfloor b + \lfloor c$

$$\Rightarrow 2^n - 1 = \lfloor b\left(1+\frac{\lfloor c}{\lfloor b}\right)$$

$\Rightarrow \lfloor b \mid 2^n - 1$ but $2^n - 1$ is odd. So $\lfloor b$ can divide 2^n-1 if and only if b=1.

So if a=1 then b=1

$\therefore 2^n = 1+1+\lfloor c$

=> $2^n - 2 = \lfloor c$

=> $2(2^{n-1}-1) = \lfloor c$

Since $2^{n-1}-1$ is odd So $2 \mid \lfloor c$ but $2^2 \nmid \lfloor c$. Since c=2 or 3

If c=2, then

$2^n = 1+1+\lfloor 2$

=> $2^n = 4$ => n=2

$\therefore$ (1, 1, 2, 2) is a soln.

If c=3, then $2^n = 1+1+\lfloor 3$

=> $2^n = 8$

=> n=3

$\therefore$ (1, 1, 3, 3) is a soln.

Next if a=2, then $2^n = 2 + \lfloor b + \lfloor c$

$$\Rightarrow 2\times\left(2^{n-1}-1\right) = \lfloor b + \lfloor c$$

$$= \lfloor b\left(1+\frac{\lfloor c}{\lfloor b}\right)$$

So, $\lfloor b \mid 2(2^{n-1}-1)$

Since $2^{n-1}-1$ is odd, so b=1 or 2 or 3.

But since $a \leq b \leq c$, So $b \neq 1$. $\lfloor c$

If b=2, then $2^n = 2+2+\lfloor c$

=> $2^n - 4 = \lfloor c$

=> $2^2(2^{n-2}-1)=\lfloor c$

=> $2^2||\lfloor c$ but $2^3||\lfloor c$

$\therefore$ c=4

$\therefore$ $2^n=2+2+\lfloor 4$

=> $2^n=2+2+24$

=> $2^n=28$

Which has no solution in integers.

If b=3, then $2^n=2+6+\lfloor c$

=> $2^n-8=\lfloor c$

=>$2^3)2^{n-3}-1)=\lfloor c$

=>$2^3||\lfloor c$ but $2^4 \nmid \lfloor c$

$\therefore$ c=4 or c=5

If c=4 then $2^n=2+6+\lfloor 4$

=>$2^n=32$

=>n=5

$\therefore$ (2, 3, 4, 5) is a soln.

If c=5, then $2^n=2+6+\lfloor 5$

=> $2^n=128$

=> n=7

$\therefore$ (2, 3, 5, 7) is a soln.

Thus, the only possible solutions are (1, 1, 2, 2), (1, 1, 3, 3,), (2, 3, 4, 5) and (2, 3, 5, 7).

13. Let $3k+2$ be a prime number and a,b be positive integers such that

$$\frac{a}{b}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots-\frac{1}{2k}+\frac{1}{2k+1}$$

Show that 3k+2 divides a.

(Hint : Group the sum in RHS into positive and negative terms. Simplify and rearrange suitably to extract 3k+2 from the sum. Then use the difinition of prime number.)

Soln.

Since 3k+2 is prims, so K is odd.

$$\frac{a}{b}=\left(1+\frac{1}{3}+\frac{1}{5}+\ldots\ldots+\frac{1}{2k+1}\right)-\left(\frac{1}{2}+\frac{1}{4}+\ldots.+\frac{1}{2k}\right)$$

$$=\left(1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{2k}+\frac{1}{2K+1}\right)-2\left(\frac{1}{2}+\frac{1}{4}+\ldots..+\frac{1}{2k}\right)$$

$$=\left(1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{2k}+\frac{1}{2K+1}\right)-\left(1+\frac{1}{2}+\frac{1}{3}+\ldots..+\frac{1}{k}\right)$$

$$=\frac{1}{K+1}+\frac{1}{K+2}+\ldots.+\frac{1}{2K}+\frac{1}{2K+1}$$

$$=\left(\frac{1}{K+1}+\frac{1}{2K+1}\right)+\left(\frac{1}{K+2}+\frac{1}{2K}\right)+\ldots.+\left(\frac{1}{K+\frac{K+1}{2}}+\frac{1}{K+\frac{K+1}{2}+1}\right)$$

($\therefore$K is odd, so K+1 is even)

$$=\frac{3K+2}{(K+1)(2K+1)}+\frac{3K+2}{(K+2)(2K}+\ldots.+\frac{3K+2}{\left(K+\frac{K+1}{2}\right)\left(K+\frac{K+1}{2}+1\right)}$$

$$=(3K+2)\left[\frac{P}{(K+1)(K+2)\ldots..(2K)\,(2K+1)}\right]$$ where P is a +ve integer.

$$\Rightarrow a(K+1)(K+2)\ldots.*\,(2K)(2K+1)=(3K+2)\text{p.b.}$$

$\therefore$ 3K+2|(LHS)

Since 3K+2 is prime, so 3K+2 divides one of the factors but none of the factors K+1, K+2, ..., 2K, 2K+1 is divisible by 3K+2, so 3K+2 divides a.

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